FINITE-DIMENSIONAL MODEL SPACES INVARIANT UNDER COMPOSITION OPERATORS

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Abstract. Finite-dimensional model spaces are quotient spaces of the Hardy space on the open unit disc, determined by finite Blaschke products. Composition operators, on the other hand, act by composing Hardy space functions with analytic self-maps of the open unit disc. Both are classical and well-studied objects in the theory of analytic function spaces. In this paper, we present a complete characterization of finite-dimensional model spaces that are invariant under composition operators. Finite cyclic groups and the prime factorizations of natural numbers play a crucial role in understanding the structure of such invariant subspaces and the associated analytic self-maps.

Contents

1. Introduction

The genesis of this paper lies in the interplay between two natural and widely studied analytic objects: composition operators and finite-dimensional spaces associated with Blaschke products. The latter class of spaces is known as model spaces (more specifically, as finite-dimensional model spaces).

Unbeknownst to Mashreghi et al. [9], the first and second authors in [11] studied this problem in broad generality, yielding abstract results and leaving many questions unresolved even at the level of finite-dimensional model spaces. In this paper, we revisit the results of [9] and [11] and present them in a more unified and broader framework at the level of finite-dimensional model spaces. The results presented in this paper provide a comprehensive treatment of the subject. This paper also identifies and corrects certain errors in [9], sharpens some of its results and proofs, and answers a question posed therein. Nevertheless, some of the groundwork and underlying notions (such as group-theoretic

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tools) employed in this paper originate from [9] (as well as from [11] to some extent). In our analysis, we combine finite group-theoretic methods with insights drawn from the prime factorizations of natural numbers, specifically those corresponding to the sizes of the zero sets of finite Blaschke products. We now proceed to introduce the key concepts of the paper.

For each λ in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we define the corresponding *Blaschke product* b_λ as follows:

$$b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z},$$

for all $z \in \mathbb{D}$. Given $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$, we define the *finite Blaschke product*

$$\theta = \prod_{j=1}^{n} b_{\lambda_j}.$$

The *model space* (or, *finite-dimensional model space* to be more specific) associated with θ is defined as

$$\mathcal{Q}_\theta = H^2 \ominus \theta H^2,$$

where H^2 denotes the Hardy space on $\mathbb{D}$. It is known that

$$\dim \mathcal{Q}_\theta = n.$$

The other key object in our study is the composition operator. Given an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$ (in short, $\varphi \in \mathcal{S}(\mathbb{D})$), the *composition operator* C_φ is defined on H^2 by

$$C_\varphi f = f \circ \varphi \qquad (f \in H^2).$$

It is a classical result that C_φ is a bounded linear operator on H^2 for all $\varphi \in \mathcal{S}(\mathbb{D})$ [13]. The goal of this paper is to determine finite Blaschke products θ and analytic self-maps $\varphi \in \mathcal{S}(\mathbb{D})$ such that

$$C_\varphi \mathcal{Q}_\theta \subseteq \mathcal{Q}_\theta,$$

That is, the finite-dimensional model spaces $\mathcal{Q}_\theta$ that are invariant under the composition operator C_φ. More specifically, our aim is to study the following object:

$$D(\mathcal{Q}_\theta) = \{\varphi \in \mathcal{S}(\mathbb{D}) : C_\varphi \mathcal{Q}_\theta \subseteq \mathcal{Q}_\theta\}.$$

Given that the elements of $\mathcal{Q}_\theta$ are rational functions, one can consider $\mathcal{Q}_\theta$ as a function space defined on the extended complex plane; that is, the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. This perspective suggests another object similar to $D(\mathcal{Q}_\theta)$, defined as follows:

$$L(\mathcal{Q}_\theta) = \{\varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty \text{ analytic}, \varphi \not\equiv \infty, \text{ and } C_\varphi \mathcal{Q}_\theta \subseteq \mathcal{Q}_\theta\}$$

Observe that rational functions are precisely the functions holomorphic at ∞; hence, every element of $L(\mathcal{Q}_\theta)$ is necessarily a rational function (also see [9, Section 1]). The motivation for these objects comes from [9] (and also [11]). Three natural questions arise immediately:

1. What are the elements of the set $D(\mathcal{Q}_\theta)$?
2. What type of structure do these elements form?
3. The same questions as (1) and (2) above, but considered in the setting of $L(\mathcal{Q}_\theta)$.

There are many other questions that arise naturally. For instance, in [9], Mashreghi et al. posed the problem of classifying finite Blaschke products θ for which

$$L(\mathcal{Q}_\theta) \neq \{z\},$$

where z denotes the identity map. Of course, the same question makes sense in the case of $D(\mathcal{Q}_\theta)$: for which Blaschke products θ does one have $D(\mathcal{Q}_\theta) \neq \{z\}$? We address all of these questions and show that the answers vary from case to case, as is typical in the theory of composition operators. In particular, as we will see, the results naturally divide into two classes: those corresponding to finite Blaschke products that vanish at the origin and those that do not. Following [9], we also discuss the group structure associated with them. As

$$z \in L(\mathcal{Q}_\theta) \cap D(\mathcal{Q}_\theta),$$

it follows that $L(\mathcal{Q}_\theta)$ and $D(\mathcal{Q}_\theta)$ are nonempty. In fact, for any finite Blaschke product θ other than rotations (see Remark 2.1 for rotations), we have (see Proposition 2.2)

$$D(\mathcal{Q}_\theta) \subseteq L(\mathcal{Q}_\theta).$$

It is easy to see that $L(\mathcal{Q}_\theta)$ and $D(\mathcal{Q}_\theta)$ are semigroups. Throughout this article, all groups and semigroups are considered with respect to function composition. We now outline some of the main results of this paper: We begin by considering the case where θ is a finite Blaschke product with a single zero of arbitrary multiplicity:

$$\theta = b_\lambda^n,$$

for some $\lambda \in \mathbb{D} \setminus \{0\}$ and $n \in \mathbb{N}$. In Proposition 3.1, we recollect from [9, Theorem 2.1] that

$$L(\mathcal{Q}_\theta) = \left\{(1 - \bar{\lambda}a)z + a : a \in \mathbb{C}, a \neq \frac{1}{\bar{\lambda}}\right\},$$

and point out that

$$D(\mathcal{Q}_\theta) = \{z\}.$$

In particular, $D(\mathcal{Q}_\theta)$ is a trivial group, whereas $L(\mathcal{Q}_\theta)$ is a noncyclic infinite group. Note also that $L(\mathcal{Q}_\theta) \subseteq \mathrm{Aut}(\mathbb{C})$ and $D(\mathcal{Q}_\theta) \subseteq \mathrm{Aut}(\mathbb{D})$. Throughout, $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{C})$ denote the sets of all biholomorphic maps of $\mathbb{D}$ and $\mathbb{C}$, respectively.

For a general finite Blaschke product θ that is nonvanishing at the origin, the following holds (see Theorem 3.5): $\varphi \in D(\mathcal{Q}_\theta)$ if and only if there exists a constant $\alpha \in \mathbb{T}$ such that $\varphi(z) = \alpha z$ with

$$mult_\theta(\lambda) = mult_\theta(\bar{\alpha}\lambda)$$

for all $\lambda \in \mathcal{Z}(\theta)$.

For an analytic function f on $X \subseteq \mathbb{C}$, we denote its zero set by

$$\mathcal{Z}(f) = \{\alpha \in X : f(\alpha) = 0\}.$$

We denote by $mult_f(\alpha)$ the multiplicity of α as a zero of f. Note that $mult_f(\alpha) = 0$ indicates that $f(\alpha) \neq 0$. The above result also should be attributed to Mashreghi et al. [9, Corollary 2.4]. However, our presentation, proof, and perspective differ slightly. This formulation is also best suited to our framework (see the discussions preceding and following Theorem 3.5). For instance, from the above, it is now clear that rotations are the appropriate candidates for the set $D(\mathcal{Q}_\theta)$.

A question arises: which subsets of $\mathbb{T}$ give rise to such a set of rotations? Moreover, how can such subsets be related to the finite Blaschke product θ? Within the same setting, we obtain the following answer to this question (see Corollary 3.6):

(1) $\alpha z \in D(\mathcal{Q}_\theta)$ if and only if $\bar{\alpha} z \in D(\mathcal{Q}_\theta)$.
(2) $\alpha z \in D(\mathcal{Q}_\theta)$ if and only if $mult_\theta(\lambda) = mult_\theta(\alpha\lambda)$ for all $\lambda \in \mathcal{Z}(\theta)$.
(3) $\alpha z \notin D(\mathcal{Q}_\theta)$ if and only if there exists an $\lambda \in \mathcal{Z}(\theta)$ such that $mult_\theta(\lambda) \neq mult_\theta(\alpha\lambda)$.

The choice of scalars α in $\mathbb{T}$ above admits a group-theoretic interpretation, which can be stated as follows (see Theorem 4.1): Assume that

$$\theta = \prod_{i=1}^{n} b_{\lambda_i}^{m_i},$$

for distinct $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{D}\setminus\{0\}$ and natural numbers m_i, $i = 1, \ldots, n$. Then $\varphi \in D(\mathcal{Q}_\theta)$ if and only if $\varphi = \omega z$, where

$$\bar{\omega} = \frac{\lambda_{\sigma(1)}}{\lambda_1} = \frac{\lambda_{\sigma(2)}}{\lambda_2} = \cdots = \frac{\lambda_{\sigma(n)}}{\lambda_n},$$

for some $\sigma \in S_n$ with $m_i = m_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$. Moreover, in this case, we have (see Corollary 4.5)

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z\rangle,$$

for some divisor d of $n = \#\mathcal{Z}(\theta)$. Here we follow the standard notation: We use $\langle e^{\frac{2\pi i}{m}} z\rangle$, $m \in \mathbb{N}$, to denote the finite cyclic group

$$\left\{e^{\frac{2\pi i t}{m}} z : t = 0, 1, \ldots, m-1\right\},$$

under composition generated by $e^{\frac{2\pi i}{m}} z$ (note that $e^{\frac{2\pi i}{m}}$ is the primitive m-th root of unity). We also denote by S_n the symmetric group on n letters.

This raises a number of natural questions, many of which have been both posed and addressed in this paper. One problem we highlight here is the following: given θ as above, under what conditions do we have

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{n}} z\rangle?$$

In Theorem 4.8, we prove that $D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{n}} z\rangle$ if and only if $\left\{\frac{\lambda_1}{\lambda_1}, \frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}\right\}$ is a multiplicative group, and

$$m_1 = m_2 = \cdots = m_n.$$

We now turn to the question of determining when $L(\mathcal{Q}_\theta) = \{z\}$ as well as $D(\mathcal{Q}_\theta) = \{z\}$. We provide the following answer (see Theorem 5.3): Let θ be a finite Blaschke product. Assume that $\theta(0) \neq 0$. Consider the prime factorization of $n := \#\mathcal{Z}(\theta)$ as

$$n = p_1^{k_1} \cdots p_m^{k_m}.$$

Then

$$D(\mathcal{Q}_\theta) = \{z\},$$

if and only if for each $j \in \{1, \ldots, m\}$, there exists $\lambda_j \in \mathcal{Z}(\theta)$ such that

$$mult_\theta(\lambda_j) \neq mult_\theta(e^{\frac{2\pi i}{p_j}} \lambda_j).$$

Moreover, in Theorem 5.4, we prove the following: Given a finite Blaschke product θ, we have $L(Q_\theta) = \{z\}$ if and only if the following conditions hold:

(1) $\theta(0) \neq 0$.
(2) $\#\mathcal{Z}(\theta) \geq 2$.
(3) For every non-constant affine map $az + b$, other than the identity, there exists $\lambda \in \mathcal{Z}(\theta)$ such that
$$mult_\theta(\lambda) \neq mult_\theta\left(\frac{\bar{a}\lambda}{1 - \bar{b}\lambda}\right).$$

The situation changes when we shift our focus to finite Blaschke products that vanish at the origin. To illustrate this, we outline the following results, as observed in Corollary 6.3: For $\lambda \in \mathbb{D} \setminus \{0\}$ and $n \geq 1$, define
$$\theta = zb_\lambda^n.$$
Then
$$L(\mathcal{Q}_\theta) = \left\{\varphi \in \mathrm{Mob}(\mathbb{C}_\infty) : \varphi\left(\frac{1}{\bar{\lambda}}\right) = \frac{1}{\bar{\lambda}}\right\} \cup \mathbb{C},$$
and
$$D(\mathcal{Q}_\theta) = \left\{\varphi \in \mathcal{S}(\mathbb{D}) \cap \mathrm{Mob}(\mathbb{C}_\infty) : \varphi\left(\frac{1}{\bar{\lambda}}\right) = \frac{1}{\bar{\lambda}}\right\} \cup \mathbb{D}.$$
These results are comparable to Theorem 2.5 and Corollary 2.6 of [9]. Moreover, in the above setting, we have new information:
$$(D(\mathcal{Q}_\theta) \setminus \mathbb{D}) \cap \mathrm{Aut}(\mathbb{D}),$$
is uncountable, and $L(\mathcal{Q}_\theta) \setminus \mathbb{C}$ is a non-cyclic infinite group. Here, $\mathrm{Mob}(\mathbb{C}_\infty)$ denotes the group of all Möbius transformations of the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

However, for a finite Blaschke product with a zero at the origin of higher multiplicity, we encounter a markedly different scenario once again (see Theorem 6.4): Let $m \geq 2$, $n \geq 1$, and let $\lambda \in \mathbb{D} \setminus \{0\}$. Define
$$\theta = z^m b_\lambda^n.$$
Then
$$L(\mathcal{Q}_\theta) = \left\{(1 - \bar{\lambda}a)z + a : a \neq \frac{1}{\bar{\lambda}}\right\} \cup \left(\mathbb{C} \setminus \left\{\frac{1}{\bar{\lambda}}\right\}\right),$$
and
$$D(\mathcal{Q}_\theta) = \{z\} \cup \mathbb{D}.$$
In particular, in this case (see Corollary 6.5), $D(\mathcal{Q}_\theta) \setminus \mathbb{D}$ is a trivial group and $L(\mathcal{Q}_\theta) \setminus \mathbb{C}$ is an uncountable group. This result holds after a major revision of Theorem 2.9 from [9].

Now we consider a finite Blaschke product θ satisfying $\theta(0) = 0$, $\theta'(0) \neq 0$, and
$$\#(\mathcal{Z}(\theta) \setminus \{0\}) \geq 2.$$
In Theorem 7.4 and Corollary 7.5, we prove that $\varphi \in L(\mathcal{Q}_\theta)$ if and only if φ is a Möbius transformation and
$$mult_\theta(\lambda) = mult_\theta(\tilde{\varphi}(\lambda)),$$
for all $\lambda \in \mathcal{Z}(\theta) \setminus \{0\}$. Further, we have
$$D(\mathcal{Q}_\theta) = L(\mathcal{Q}_\theta) \cap \mathcal{S}(\mathbb{D}).$$
In the case of $\varphi \in D(\mathcal{Q}_\theta)$, if φ is non-constant, then we have

(1) $\varphi \in \text{Aut}(\mathbb{D})$.
(2) There exists $n \in \mathbb{N}$ such that $\underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}} = I$.

The most general result concerning finite Blaschke products θ that vanish at the origin is Corollary 7.6, which states that if

$$\#(\mathcal{Z}(\theta) \setminus \{0\}) \geq 2,$$

then $D(\mathcal{Q}_\theta) \setminus \mathbb{C}$ forms a group under composition.

Needless to say, the above collection of results varies significantly from case to case, highlighting the rich and intricate structure of composition operators, even within the framework of finite-dimensional model spaces. Many additional results in this paper as well as in the broader literature further explore these themes along similar lines.

From this perspective, we remind the reader that the invariant subspace problem for composition operators is a classical and challenging problem. In fact, the invariant subspace problem for operators on Hilbert spaces is equivalent to the one-dimensional minimal invariant subspace problem for composition operators with hyperbolic symbols [12]. This scenario also serves as an additional motivation besides the study of lattice structures of composition operators for the theory developed in this paper. We refer the reader to [1, 3, 6, 8, 10, 11] and references therein for more results in this direction. For a list of exotic properties and their connections to diverse aspects of composition operators, we refer the reader to [2, 5, 4].

The remaining part of the paper is structured as follows. Section 2 presents some general observations that are used throughout the paper. Section 3 provides a precise description of model spaces corresponding to finite Blaschke products that do not vanish at the origin. Section 4 explores the natural emergence of finite cyclic groups, particularly in the context of finite subsets of the unit circle $\mathbb{T}$. In Section 5, we present a complete solution to the question of the nontriviality of the groups that arise naturally in the study of self-analytic functions on $\mathbb{D}$ within the framework of model spaces. Sections 6 and 7 deal with the analysis of model spaces associated with finite Blaschke products that vanish at the origin. This setting brings Möbius transformations into consideration. Section 8 is devoted entirely to a detailed example, motivated by earlier work in [9]. We revisit this example to correct certain inaccuracies and to present its full significance within a more general framework. The final section, Section 9, presents general observations, outlines potential directions for future research, and includes a summary table highlighting some of the key results obtained in this paper.

2. Basic observations

We treat this section as a warm-up for the results presented in the forthcoming sections. We also derive some elementary observations. Recall that

$$\dim \mathcal{Q}_\theta = \deg \theta,$$

for each finite Blaschke product θ. We first consider the case of one-dimensional model spaces. These spaces correspond to $\theta = b_\alpha$, $\alpha \in \mathbb{D}$. However, here we focus only on the case $\alpha = 0$ (see Proposition 3.1 for the $\alpha \neq 0$ case):

Remark 2.1. *Let $\theta = \alpha z$ for some $\alpha \in \mathbb{T}$. Then*

$$\mathcal{Q}_\theta = H^2 \ominus zH^2 = \mathbb{C},$$

where $\mathbb{C}$ represents the space of all constant functions on $\mathbb{D}$. In this case, it is trivial to note that

$$L(\mathcal{Q}_\theta) = \{\textit{all the rational functions}\},$$

and

$$D(\mathcal{Q}_\theta) = \mathcal{S}(\mathbb{D}),$$

and consequently, $L(\mathcal{Q}_\theta)$ and $D(\mathcal{Q}_\theta)$ are not comparable semigroups.

We recall a general fact about the basis vectors of finite-dimensional model spaces, which will be used frequently in what follows. Recall that if θ is a finite Blaschke product with distinct zeros $\lambda_1, \ldots, \lambda_k$ of multiplicities $n_1, \ldots, n_k$, respectively, then the set

$$\left\{ c_{\lambda_i}^{(\ell_i)} : 0 \leq \ell_i \leq n_i - 1, 1 \leq i \leq k \right\}, \tag{2.1}$$

form a basis for $\mathcal{Q}_\theta$. Here, the function $c_\lambda^{(s)}$ is defined by

$$c_\lambda^{(s)}(z) = \frac{z^s}{(1 - \bar{\lambda} z)^{s+1}}.$$

If $\lambda \neq 0$, then $c_\lambda^{(s)}$ may also be taken as

$$c_\lambda^{(s)}(z) = \frac{1}{(1 - \bar{\lambda} z)^{s+1}}. \tag{2.2}$$

In particular, we have

$$\dim \mathcal{Q}_\theta = n_1 + \cdots + n_k.$$

The following result is key, relying on the fact that $\mathcal{Q}_\theta$ consists of rational functions. While the containment is elementary, it will be useful in what follows.

Proposition 2.2. *Let θ be a finite Blaschke product that is not a rotation. Then*

$$D(\mathcal{Q}_\theta) \subseteq L(\mathcal{Q}_\theta).$$

Proof. Suppose $\varphi \in D(\mathcal{Q}_\theta)$. To show that $\varphi \in L(\mathcal{Q}_\theta)$, it suffices to show that φ is a rational map. First, assume that $z \in \mathcal{Q}_\theta$. Then

$$z \circ \varphi = \varphi \in \mathcal{Q}_\theta,$$

which implies that φ is a rational function. Next, assume that $z \notin \mathcal{Q}_\theta$. Since θ is not a rotation, it follows that

$$\mathcal{Z}(\theta) \setminus \{0\} \neq \emptyset.$$

Pick $\lambda \in \mathcal{Z}(\theta) \setminus \{0\}$. This implies $\theta = b_\lambda \tilde{\theta}$ for some finite Blaschke product $\tilde{\theta}$. Then $\dfrac{1}{1 - \bar{\lambda} z} \in \mathcal{Q}_\theta$, and hence

$$\frac{1}{1 - \bar{\lambda} z} \circ \varphi = \frac{1}{1 - \bar{\lambda} \varphi} \in \mathcal{Q}_\theta,$$

it follows, in this case as well, that φ is a rational map. Therefore, in both cases we have $\varphi \in L(\mathcal{Q}_\theta)$. This completes the proof. □

The following simple lemma will be used throughout this paper repeatedly.

Lemma 2.3. *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an analytic function. Suppose*

$$\frac{1}{1 - p\varphi(z)} = \frac{a}{1 - qz} \qquad (z \in \mathbb{D}),$$

for some complex numbers $0 < |p| \leq |q| < 1$ and $a \neq 0$. Then the following conditions are equivalent:

(1) $\varphi \in \mathcal{S}(\mathbb{D})$.
(2) $a = 1$ and $|p| = |q|$.
(3) φ is a rotation.

In either of these cases, we have $\varphi(z) = \frac{q}{p}z$.

Proof. By cross-multiplying the equation given in the hypothesis, we immediately obtain

$$\varphi(z) = \frac{q}{ap}z + \frac{a-1}{ap}.$$

Suppose (1) holds, that is, $\varphi \in \mathcal{S}(\mathbb{D})$. Assume, if possible, that $a \neq 1$. Since $0 < |p| < 1$, it follows that

$$\left|\frac{a-1}{ap}\right| > \left|\frac{a-1}{a}\right|.$$

By the triangle inequality, we have

$$\left|\frac{a-1}{ap}\right| + \left|\frac{1}{a}\frac{q}{p}\right| > \left|\frac{a-1}{a}\right| + \left|\frac{1}{a}\right| \geq 1.$$

Therefore φ is not a self-map of $\mathbb{D}$. It yields that $a = 1$, and thus

$$\varphi(z) = \frac{q}{p}z.$$

Also note that φ cannot be a self-map of $\mathbb{D}$ if $|p| < |q|$. Hence $\varphi(z) = \frac{q}{p}z$ with $|p| = |q|$. Thus, we have shown that (1) implies (2), and (2) implies (3). The implication (3) $\Rightarrow$ (1) is immediate. ☐

3. Blaschke products nonvanishing at 0

This section discusses the structures of $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$ under the condition that θ is a finite Blaschke product and that $\theta(0)$ is nonzero. Part of this section also recalls a collection of results, primarily from [9] (and also from [11]), to provide a complete structures of $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$ associated with θ. However, we note that, along the way, we will also refine some of those results to provide a clearer understanding of these sets.

We first consider the case where θ is a finite Blaschke product with a singleton zero set. A major part of the following result was established in [9] and [11]. The second part follows easily from Lemma 2.3. We will use this result in the later sections of the paper.

Proposition 3.1. *Let $\theta = b_\lambda^n$ for some $\lambda \in \mathbb{D} \setminus \{0\}$ and $n \in \mathbb{N}$. Then*

$$L(\mathcal{Q}_\theta) = \left\{(1 - \bar{\lambda}a)z + a : a \in \mathbb{C}, a \neq \frac{1}{\bar{\lambda}}\right\},$$

and

$$D(\mathcal{Q}_\theta) = \{z\}.$$

Proof. For the first part, see [9, Theorem 2.1] (or see [11, Theorem 3.1]). For the representation of $D(\mathcal{Q}_\theta)$, we apply [11, Theorem 3.1] (or [9, Theorem 2.1]) to find a constant c such that

$$1-\bar{\lambda}\varphi = c(1-\bar{\lambda}z).$$

By Lemma 2.3, we get $\varphi(z)=z$, which proves that $D(\mathcal{Q}_\theta)=\{z\}$. □

In particular, we obtain the following contrasting result:

Corollary 3.2. *Let $\theta = b_\lambda^n$ for some $\lambda \in \mathbb{D}\setminus\{0\}$ and $n\in\mathbb{N}$. Then $D(\mathcal{Q}_\theta)$ is a trivial group, whereas $L(\mathcal{Q}_\theta)$ is a noncyclic infinite group.*

This completes the discussion of finite Blaschke products with a singleton zero set. We now turn to the case of finite Blaschke products θ where $\theta(0)\neq 0$ and

$$\#\mathcal{Z}(\theta)\geq 2.$$

The following lemma is an improvement of [9, Lemma 2.2]. Specifically, we remove the assumption that φ is a rational function, which was required in the original statement. We do, however, utilize that lemma to produce the subsequent sharper version, wherein all relevant conditions are essentially consolidated into a single, unified condition.

Lemma 3.3. *Let θ be a finite Blaschke product that does not vanish at the origin. Assume that $\#\mathcal{Z}(\theta)\geq 2$. Then $\varphi\in L(Q_\theta)$ if and only if there exist constants $a(\neq 0)$ and b such that*

$$\varphi(z)=az+b,$$

with

$$mult_\theta(\lambda) = mult_\theta\left(\frac{\bar{a}\lambda}{1-\bar{b}\lambda}\right),$$

for all $\lambda\in\mathcal{Z}(\theta)$.

Proof. Suppose $\varphi\in L(Q_\theta)$. As in the proof of Proposition 2.2, φ is a rational function. Now if $\varphi = z$, then the desired result follows. On the other hand, if $\varphi\neq z$, then [9, Lemma 2.2] implies the result. For the converse direction, suppose there exist constants $a(\neq 0)$ and b such that

$$\varphi(z)=az+b,$$

with

$$\mathrm{mult}_\theta(\lambda) = \mathrm{mult}_\theta\left(\frac{\bar{a}\lambda}{1-\bar{b}\lambda}\right),$$

for all $\lambda\in\mathcal{Z}(\theta)$. To show $\varphi\in L(Q_\theta)$ (that is $C_\varphi\mathcal{Q}_\theta\subseteq\mathcal{Q}_\theta$), we start with a basis element $\frac{1}{(1-\bar{\lambda}z)^s}$, where $\lambda\in\mathcal{Z}(\theta)$ and $s\leq mult_\theta(\lambda)$. Then

$$\frac{1}{(1-\bar{\lambda}z)^s}\circ\varphi = \frac{1}{(1-\bar{\lambda}(az+b))^s} = \frac{1}{(1-\bar{b}\lambda)^s}\frac{1}{\left(1-\overline{\left(\frac{\bar{a}\lambda}{1-\bar{b}\lambda}\right)}\,z\right)^s}.$$

Combined with the fact that

$$s\leq \mathrm{mult}_\theta(\lambda) = \mathrm{mult}_\theta\left(\frac{\bar{a}\lambda}{1-\bar{b}\lambda}\right),$$

we obtain

$$\frac{1}{(1-\bar{\lambda}z)^s}\circ\varphi\in\mathcal{Q}_\theta,$$

which completes the result. ☐

The above multiplicity condition also ensures that $\frac{\bar{a}\lambda}{1-\bar{b}\lambda}\in\mathcal{Z}(\theta)$. Next, we recall from [9, Theorem 2.3] a rigidity-type result concerning affine transformation symbols for model spaces associated with Blaschke products having two distinct zeros.

Proposition 3.4. *Let θ be a finite Blaschke product. Assume that $\theta(0)\neq 0$ and $\#\mathcal{Z}(\theta)\geq 2$. Then there exist $a,b\in\mathbb{C}$ and $n\in\mathbb{N}$ such that*

$$L(\mathcal{Q}_\theta)=\left\{z,\varphi,\varphi^{[2]},\dots,\varphi^{[n-1]}\right\}$$

where

$$\varphi(z)=az+b,$$

and

$$\varphi^{[n]}=z.$$

In particular, $L(\mathcal{Q}_\theta)$ is a finite cyclic subgroup of $\operatorname{Aut}(\mathbb{C})$.

In these circumstances, we also have a description of $D(\mathcal{Q}_\theta)$ from [9, Corollary 2.4]: Let θ be a finite Blaschke product. Suppose $\theta(0)\neq 0$ and $\#\mathcal{Z}(\theta)\geq 2$. Then $\varphi\in D(\mathcal{Q}_\theta)$ if and only if following conditions hold:

(i) $\varphi(z)=\alpha z$ for some $\alpha\in\mathbb{T}$.
(ii) $\alpha^n=1$ for some $n\in\mathbb{N}$.
(iii) $\alpha\lambda\in\mathcal{Z}(\theta)$ for all $\lambda\in\mathcal{Z}(\theta)$.
(iv) The zeros $\{\lambda,\alpha\lambda,\dots,\alpha^{n-1}\lambda\}$ of θ have same multiplicity.

The proof of this result is involved, as it relies on non-trivial results concerning the iterative behavior of loxodromic and parabolic Möbius transformations. In the following, we present a slightly modified version of [9, Corollary 2.4] analogous to a conjugation of the scalar part in rotation maps along with a different proof. This version leads to several useful consequences. Moreover, it will both imply and be implied by [9, Corollary 2.4], as we will point out after the proof.

Theorem 3.5. *Let θ be a finite Blaschke product. Assume that $\theta(0)\neq 0$. Then $\varphi\in D(\mathcal{Q}_\theta)$ if and only if $\varphi(z)=\alpha z$ for some $\alpha\in\mathbb{T}$ with*

$$mult_\theta(\lambda)=mult_\theta(\bar{\alpha}\lambda)$$

for all $\lambda\in\mathcal{Z}(\theta)$.

Proof. Suppose $\theta=\prod_{i=1}^{n}b_{\lambda_i}^{m_i}$, where λ_i are nonzero distinct elements in $\mathbb{D}$ and m_i are natural numbers. Suppose that $\varphi\in D(\mathcal{Q}_\theta)$. If $n=1$, then the result simply follows from Proposition 3.1 (with the choice of $\alpha=1$). Assume that $n\geq 2$. By Proposition 3.4, there exist constants α and β such that

$$\varphi(z)=\alpha z+\beta.$$

Choose $\lambda \in \mathcal{Z}(\theta)$ such that $|\lambda| = \min\{|\lambda_1|, \dots, |\lambda_n|\}$. As $\dfrac{1}{1-\bar{\lambda}z} \in \mathcal{Q}_\theta$, we have

$$\frac{1}{1-\bar{\lambda}z} \circ \varphi = \frac{1}{1-\bar{\lambda}\varphi} = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{c_{ij}}{(1-\bar{\lambda}_i z)^j},$$

for some constants c_{ij} with at least one of them is nonzero. By the identity theorem, the equality holds on the entire complex plane except possibly at finitely many poles. Since the function on the left-hand side has exactly one pole of order 1, the equation implies that

$$\frac{1}{1-\bar{\lambda}\varphi} = \frac{c}{1-\bar{\lambda}_k z},$$

for some $k \in \{1, \dots, n\}$ and $c \neq 0$. At this point, Lemma 2.3 applies and yields

$$\varphi(z) = \alpha z,$$

for some constant $\alpha \in \mathbb{T}$. Fix $i \in \{1, \dots, n\}$. As $\frac{1}{(1-\bar{\lambda}_i z)^{m_i}} \in \mathcal{Q}_\theta$, we have

$$\frac{1}{(1-\bar{\lambda}_i \alpha z)^{m_i}} = \frac{1}{(1-\bar{\lambda}_i z)^{m_i}} \circ \varphi \in \mathcal{Q}_\theta,$$

and hence $\bar{\alpha}\lambda_i \in \mathcal{Z}(\theta)$. Indeed, if $\bar{\alpha}\lambda_i \notin \mathcal{Z}(\theta)$, then, by considering a basis for $\mathcal{Q}_\theta$ consisting of functions of the form given in (2.2), we obtain

$$\frac{1}{(1-\bar{\lambda}_i \alpha z)^{m_i}} \notin \mathcal{Q}_\theta,$$

leading to a contradiction. Thus, we have that $\lambda \in \mathcal{Z}(\theta)$ implies $\bar{\alpha}\lambda \in \mathcal{Z}(\theta)$. This also implies that

$$m_i = mult_\theta(\lambda_i) \leq mult_\theta(\bar{\alpha}\lambda_i).$$

By applying this argument iteratively, we conclude, for all $\lambda \in \mathcal{Z}(\theta)$ and $k \in \mathbb{N}$, that

$$\bar{\alpha}^k \lambda \in \mathcal{Z}(\theta),$$

and

$$mult_\theta(\lambda) \leq mult_\theta(\bar{\alpha}\lambda) \leq \cdots \leq mult_\theta(\bar{\alpha}^k \lambda).$$

Moreover, as θ has only finitely many zeros, it follows that

$$\bar{\alpha}^m = 1, \tag{3.1}$$

for some $m \in \mathbb{N}$, and then, for any $\lambda \in \mathcal{Z}(\theta)$, we have

$$mult_\theta(\lambda) \leq mult_\theta(\bar{\alpha}\lambda) \leq \cdots \leq mult_\theta(\bar{\alpha}^m \lambda) = mult_\theta(\lambda),$$

and hence, $mult_\theta(\lambda) = mult_\theta(\bar{\alpha}\lambda)$.
The converse part is easy: Suppose α is a constant such that $\varphi(z) = \alpha z$ and $mult_\theta(\lambda) = mult_\theta(\bar{\alpha}\lambda)$ for all $\lambda \in \mathcal{Z}(\theta)$. Then for any arbitrary basis element $\frac{1}{(1-\bar{\lambda}z)^l}$ of $\mathcal{Q}_\theta$, for some $l \in \{1, \dots, mult_\theta(\lambda)\}$, we have

$$\frac{1}{(1-\bar{\lambda}z)^l} \circ \varphi = \frac{1}{(1-\bar{\lambda}\alpha z)^l} \in \mathcal{Q}_\theta,$$

and hence $\varphi \in D(\mathcal{Q}_\theta)$. This completes the proof of the theorem. □

It is important to note that $\bar{\alpha}^m = 1$ for some $m \in \mathbb{N}$ and $mult_\theta(\lambda) = mult_\theta(\bar{\alpha}\lambda)$ for all $\lambda \in \mathcal{Z}(\theta)$ in the above result is equivalent to

$$mult_\theta(\lambda) = mult_\theta(\alpha\lambda),$$

for all $\lambda \in \mathcal{Z}(\theta)$. This recovers the exact statement of [9, Corollary 2.4]. Similarly, it also recovers the version we proved above. As an immediate consequence of these results, we have the following:

Corollary 3.6. *Let θ be a finite Blaschke product and let $\alpha \in \mathbb{T}$. Assume that $\theta(0) \neq 0$. Then we have the following:*

(1) $\alpha z \in D(\mathcal{Q}_\theta)$ if and only if $\bar{\alpha} z \in D(\mathcal{Q}_\theta)$.
(2) $\alpha z \in D(\mathcal{Q}_\theta)$ if and only if $mult_\theta(\lambda) = mult_\theta(\alpha\lambda)$ for all $\lambda \in \mathcal{Z}(\theta)$.
(3) $\alpha z \notin D(\mathcal{Q}_\theta)$ if and only if there exists an $\lambda \in \mathcal{Z}(\theta)$ such that $mult_\theta(\lambda) \neq mult_\theta(\alpha\lambda)$.

Results concerning representations of functions in $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$, corresponding to finite Blaschke products θ that vanish at the origin, will be considered in Section 6.

4. Cyclic groups

Here, we continue with the setting of the previous section and introduce group-theoretic tools to study the sets $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$.

In the setting described in Corollary 3.6, we know that α is of finite order-specifically,

$$\alpha^m = 1,$$

as observed in equation (3.1) (and also in [9, Corollary 2.4], as previously noted). We now turn to providing a precise interpretation of the index m and its role in the structure of $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$. This is done through the lens of finite cyclic groups. To that end, we first present another characterization of functions in the class $D(\mathcal{Q}_\theta)$. For each $n \in \mathbb{N}$, in what follows, we write

$$J_n = \{1, \ldots, n\},$$

and denote by S_n the symmetric group of degree n.

Theorem 4.1. *Let $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{D} \setminus \{0\}$ be a set of distinct elements and let $m_i \in \mathbb{N}$, $i = 1, \ldots, n$. Let $\theta = \prod_{i=1}^{n} b_{\lambda_i}^{m_i}$. Then $\varphi \in D(\mathcal{Q}_\theta)$ if and only if $\varphi = \omega z$, where*

$$\bar{\omega} = \frac{\lambda_{\sigma(1)}}{\lambda_1} = \frac{\lambda_{\sigma(2)}}{\lambda_2} = \cdots = \frac{\lambda_{\sigma(n)}}{\lambda_n},$$

for some $\sigma \in S_n$ with $m_i = m_{\sigma(i)}$ for all $i \in J_n$.

Proof. Let $\varphi \in D(\mathcal{Q}_\theta)$. By Theorem 3.5, there exists $\omega \in \mathbb{T}$ such that $\varphi(z) = \omega z$ and $\bar{\omega}\lambda \in \mathcal{Z}(\theta)$ for all $\lambda \in \mathcal{Z}(\theta)$. Thus for each $i \in J_n$, there is an $j \in J_n$ such that $\bar{\omega}\lambda_i = \lambda_j$. Define $\sigma : J \to J$ by

$$\overline{\omega}\lambda_i = \lambda_{\sigma(i)},$$

for all $i \in J_n$. It is easy to see that σ is injective. Since J_n is finite set, σ is onto. Thus $\sigma \in S_n$ and

$$\overline{\omega} = \frac{\lambda_{\sigma(1)}}{\lambda_1} = \frac{\lambda_{\sigma(2)}}{\lambda_2} = \cdots = \frac{\lambda_{\sigma(n)}}{\lambda_n}.$$

Since $\omega z \in D(\mathcal{Q}_\theta)$, again by Theorem 3.5, for each $i \in J_n$, we have

$$m_i = mult_\theta(\lambda_i) = mult_\theta(\overline{\omega}\lambda_i) = mult_\theta(\lambda_{\sigma(i)}) = m_{\sigma(i)}.$$

For the converse direction, assume $\varphi = \omega z$, where ω satisfies the identity given in the statement. Then

$$C_\varphi\left(\frac{1}{(1-\bar{\lambda}_i z)^k}\right) = \frac{1}{(1-\bar{\lambda}_i z)^k} \circ \varphi = \frac{1}{(1-\bar{\lambda}_i \omega z)^k} = \frac{1}{(1-\bar{\lambda}_{\sigma(i)} z)^k} \in \mathcal{Q}_\theta,$$

for all $k \in \{1, \dots, m_i\}$ and $i \in J_n$. Hence $\varphi \in D(\mathcal{Q}_\theta)$, completing the proof of the theorem. □

Remark 4.2. *In the setting of Theorem 4.1, pick $\omega z \in D(\mathcal{Q}_\theta)$ and $\sigma \in S_n$. Clearly*

$$\omega^n = \frac{\overline{\lambda_{\sigma(1)}}}{\overline{\lambda_1}} \cdot \frac{\overline{\lambda_{\sigma(2)}}}{\overline{\lambda_2}} \cdots \frac{\overline{\lambda_{\sigma(n)}}}{\overline{\lambda_n}} = 1.$$

Moreover, if σ contains a cycle $(i_1 i_2 \dots i_k)$, that is, if $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \dots, \sigma(i_{k-1}) = i_k$, and $\sigma(i_k) = i_1$, then

$$\omega^k = \frac{\overline{\lambda_{\sigma(i_1)}}}{\overline{\lambda_{i_1}}} \cdot \frac{\overline{\lambda_{\sigma(i_2)}}}{\overline{\lambda_{i_2}}} \cdots \frac{\overline{\lambda_{\sigma(i_k)}}}{\overline{\lambda_{i_k}}} = 1.$$

Recall that for each $n \in \mathbb{N}$, we use $\langle e^{\frac{2\pi i}{n}} z\rangle$ to denote the finite cyclic group

$$\left\{ e^{\frac{2\pi i t}{n}} z : t = 0, 1, \dots, n-1 \right\},$$

under composition generated by $e^{\frac{2\pi i}{n}} z$, where $e^{\frac{2\pi i}{n}}$ is the primitive n-th root of unity. The above remark yields the following corollary:

Corollary 4.3. *Let θ be a finite Blaschke product with $\theta(0) \neq 0$. If $n := \#\mathcal{Z}(\theta)$, then*

$$D(\mathcal{Q}_\theta) \subseteq \langle e^{\frac{2\pi i}{n}} z\rangle.$$

We can say a little more about $D(\mathcal{Q}_\theta)$:

Lemma 4.4. *Let θ be a finite Blaschke product with $\theta(0) \neq 0$. If $n := \#\mathcal{Z}(\theta)$, then $D(\mathcal{Q}_\theta)$ is a cyclic subgroup of $\langle e^{\frac{2\pi i}{n}} z\rangle$.*

Proof. Let $\varphi \in D(\mathcal{Q}_\theta)$. By Theorem 3.5, there exists $\omega \in \mathbb{T}$ such that $\varphi = \omega z$. By part (1) of Corollary 3.6, we have

$$\frac{1}{\omega} z = \overline{\omega} z \in D(\mathcal{Q}_\theta).$$

Note that $\overline{\omega} z$ is the inverse of ωz under composition. Hence $D(\mathcal{Q}_\theta)$ is a subgroup of the cyclic group $\langle e^{\frac{2\pi i}{n}} z\rangle$. □

We now recall the general fact that the order of any subgroup of a finite cyclic group divides the order of the group. Therefore, we obtain the following corollary (this result should be comparable to [9, Theorem 2.3]):

Corollary 4.5. *Let θ be a finite Blaschke product. Assume that $\theta(0) \neq 0$. Then*

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z\rangle,$$

for some divisor d of $\#\mathcal{Z}(\theta)$.

As an example, we consider the following:

Example 4.6. *Suppose θ is a finite Blaschke product with four distinct zeros and $\theta(0) \neq 0$. Then there are exactly three possible cyclic groups for $D(\mathcal{Q}_\theta)$:*

$$\{z\}, \langle -z \rangle = \{z, -z\}, \text{ and } \langle iz \rangle = \{z, -z, iz, -iz\}.$$

Given an integer n and a divisor d of n, one expects that there exists a finite Blaschke product θ such that $\#\mathcal{Z}(\theta) = n$ and $D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z \rangle$. This is indeed the case:

Theorem 4.7. *Let $n \in \mathbb{N}$, and let d be a divisor of n. Then there exists a Blaschke product θ such that $n := \#\mathcal{Z}(\theta)$ and*

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z \rangle.$$

Proof. Let $md = n$. Fix scalars $0 < r_1 < \cdots < r_m < 1$, and consider the finite Blaschke product

$$\theta = \prod_{t=1}^{m} \prod_{s=1}^{d} b_{\alpha^s r_t}^t,$$

where $\alpha = e^{\frac{2\pi i}{d}}$. Clearly, $\#\mathcal{Z}(\theta) = n$, and

$$\mathcal{Z}(\theta) = \{\alpha^s r_t : 1 \leq s \leq d, 1 \leq t \leq m\},$$

and

$$mult_\theta(\alpha^s r_t) = t,$$

for all $1 \leq s \leq k$. In particular, θ has exactly $n (= dm)$ zeros, all located on the circle $|z| = r_j$, for each $j = 1, \ldots, m$. Label all zeros of θ as $\lambda_1, \ldots, \lambda_n$ so that

$$\lambda_1 = r_1 \alpha, \ldots, \lambda_d = r_1 \alpha^d,$$

and then

$$\lambda_{d+1} = r_2 \alpha, \ldots, \lambda_{2d} = r_2 \alpha^d,$$

and so on. Accordingly, the first d zeros have multiplicity 1, the next d zeros have multiplicity 2, and so forth. Suppose $\varphi \in D(\mathcal{Q}_\theta)$, that is, $C_\varphi(\mathcal{Q}_\theta) \subseteq \mathcal{Q}_\theta$. By Theorem 4.1, there exist $\beta \in \mathbb{T}$ and $\sigma \in S_n$ such that $\varphi(z) = \beta z$ and

$$\bar{\beta} = \frac{\lambda_{\sigma(i)}}{\lambda_i},$$

and $mult_\theta(\lambda_i) = mult_\theta(\lambda_{\sigma(i)})$ for all $i = 1, \ldots, n$. In particular, this implies that the restriction of σ acts as a permutation on each of the sets $\{1, \ldots, d\}$, $\{d+1, \ldots, 2d\}$, and so on. In particular,

$$\bar{\beta}^d = \frac{\lambda_{\sigma(1)}}{\lambda_1} \frac{\lambda_{\sigma(2)}}{\lambda_2} \cdots \frac{\lambda_{\sigma(d)}}{\lambda_d} = 1,$$

that is, $\beta^d = 1$. As $\beta \in \left\{1, \alpha, \ldots, \alpha^{d-1}\right\}$, it follows that $D(\mathcal{Q}_\theta) \subseteq \langle \alpha z \rangle$. For the reverse inclusion, note that from our construction it is clear that

$$mult_\theta(\lambda) = mult_\theta(\alpha \lambda),$$

for all $\lambda \in \mathcal{Z}(\theta)$. Then, by part (2) of Corollary 3.6, it follows that $\alpha z \in D(\mathcal{Q}_\theta)$. Since $D(\mathcal{Q}_\theta)$ is a semi-group, we conclude that

$$\{\alpha z, \alpha^2 z, \ldots, \alpha^{d-1} z, \alpha^d z = z\} \subseteq D(\mathcal{Q}_\theta),$$

and hence $\langle \alpha z\rangle \subseteq D(\mathcal{Q}_\theta)$. This completes the proof. $\square$

Given a finite Blaschke product θ with $\theta(0) \neq 0$, by Corollary 4.3, we know that $D(\mathcal{Q}_\theta) \subseteq \langle e^{\frac{2\pi i}{n}} z\rangle$, where $\#\mathcal{Z}(\theta) = n$. It is natural to ask, when do we have

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{n}} z\rangle?$$

In the following, we answer to this question. Before that, we set up a notation: Given a set of n distinct points $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{D} \setminus \{0\}$, we define

$$\Lambda_n = \left\{\frac{\lambda_1}{\lambda_1}, \frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}\right\}.$$

Theorem 4.8. *Let $\{\lambda_1, \ldots, \lambda_n\}$ be a set of n distinct points in $\mathbb{D} \setminus \{0\}$ and let $m_i \in \mathbb{N}$, $i = 1, \ldots, n$. Define*

$$\theta = \prod_{i=1}^{n} b_{\lambda_i}^{m_i}.$$

Then

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{n}} z\rangle,$$

if and only if Λ_n is a multiplicative group, and

$$m_1 = m_2 = \cdots = m_n.$$

Proof. Let $\alpha = e^{\frac{2\pi i}{n}}$. First, we prove the sufficiency part. Since all the λ_i's are distinct, the multiplicative group Λ_n has order n. Consequently, every element of Λ_n must be an n-th root of unity. This implies that

$$\Lambda_n = \left\{1, \alpha, \ldots, \alpha^{n-1}\right\},$$

and hence

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{\lambda_1, \alpha\lambda_1, \ldots, \alpha^{n-1}\lambda_1\}.$$

By Corollary 3.6, we know that $e^{\frac{2\pi i}{n}} z \in D(\mathcal{Q}_\theta)$, which implies $\langle e^{\frac{2\pi i}{n}} z\rangle \subseteq D(\mathcal{Q}_\theta)$. The reverse inclusion, $D(\mathcal{Q}_\theta) \subseteq \langle e^{\frac{2\pi i}{n}} z\rangle$, is due to Corollary 4.3. This proves that $\langle e^{\frac{2\pi i}{n}} z\rangle = D(\mathcal{Q}_\theta)$. For the necessary part, assume that $D(\mathcal{Q}_\theta) = \langle \alpha z\rangle$. As $\alpha z \in D(\mathcal{Q}_\theta)$, by Corollary 3.6, we have

$$\{\alpha\lambda_1, \ldots, \alpha^{n-1}\lambda_1\} \subseteq \mathcal{Z}(\theta),$$

and the multiplicities of all these zeros are the same as that of λ_1. Consequently, $\Lambda_n = \{1, \alpha, \ldots, \alpha^{n-1}\}$, which completes the proof of the theorem. $\square$

We know from Corollary 4.5 that $D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z\rangle$ for some divisor d of n. This leads us to the following question: Given a finite Blaschke product θ not vanishing at the origin and a divisor d of $n := \#\mathcal{Z}(\theta)$, when does

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z\rangle?$$

The following result provides an answer. Here, we make use of the prime factorization of $n = \#\mathcal{Z}(\theta)$.

Theorem 4.9. *Let θ be a finite Blaschke product and let d be a divisor of $n := \#\mathcal{Z}(\theta)$. Suppose $\theta(0) \neq 0$ and let*

$$n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m},$$

be the prime factorization of n. Then

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z \rangle,$$

if and only if

$$mult_\theta(\lambda) = mult_\theta(e^{\frac{2\pi i}{d}} \lambda),$$

for all $\lambda \in \mathcal{Z}(\theta)$, and for each $j \in \{1, \dots, m\}$, there exist $\lambda_j \in \mathcal{Z}(\theta)$ such that

$$mult_\theta(\lambda_j) \neq mult_\theta(e^{\frac{2\pi i}{d p_j}} \lambda_j).$$

Proof. Suppose $D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z \rangle$. Then $e^{\frac{2\pi i}{d}} z \in D(\mathcal{Q}_\theta)$ and $e^{\frac{2\pi i}{d p_j}} z \notin D(\mathcal{Q}_\theta)$ for all $j = 1, \dots, m$. The necessary part then follows from Theorem 3.5 and Corollary 3.6. For the reverse direction, assume that $mult_\theta(\lambda) = mult_\theta(e^{\frac{2\pi i}{d}} \lambda)$ for all $\lambda \in \mathcal{Z}(\theta)$. Theorem 3.5 implies that

$$e^{\frac{2\pi i}{d}} z \in D(\mathcal{Q}_\theta),$$

and hence $\langle e^{\frac{2\pi i}{d}} z \rangle \subseteq D(\mathcal{Q}_\theta)$. On the other hand, by Theorem 4.5, there exists a divisor ℓ of n such that

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{\ell}} z \rangle.$$

Suppose $\ell \neq d$. Since $e^{\frac{2\pi i}{d}} z \in \langle e^{\frac{2\pi i}{\ell}} z \rangle$, there exist $k \in \{1, \dots, \ell\}$ such that

$$e^{\frac{2\pi i}{d}} = e^{\frac{2\pi i k}{\ell}},$$

which implies

$$\frac{1}{d} - \frac{k}{\ell} \in \mathbb{Z}.$$

As $d \geq 1$ and $\frac{k}{\ell} \in (0, 1]$, it follows that

$$\frac{1}{d} - \frac{k}{\ell} \in (-1, 1),$$

which yields

$$\frac{1}{d} - \frac{k}{\ell} = 0,$$

equivalently, $\ell = kd$. In particular, d is a divisor of ℓ. Since $\ell \neq d$ and $p_1, \dots, p_m$ are all the prime factors of ℓ, it follows that

$$d p_j | \ell,$$

for some $j = 1, \dots, m$. Then

$$e^{\frac{2\pi i}{d p_j}} z \in \langle e^{\frac{2\pi i}{\ell}} z \rangle = D(\mathcal{Q}_\theta),$$

and hence, by Theorem 3.5, we have $mult_\theta(\lambda) = mult_\theta(e^{\frac{2\pi i}{d p_j}} \lambda)$ for all $\lambda \in \mathcal{Z}(\theta)$. This contradicts to the other condition that for each $j \in \{1, \dots, m\}$, there exist $\lambda_j \in \mathcal{Z}(\theta)$ such that

$$mult_\theta(\lambda_j) \neq mult_\theta(e^{\frac{2\pi i}{d p_j}} \lambda_j).$$

Therefore, we conclude that $\ell = d$, that is, $D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z \rangle$. This completes the proof of the theorem. $\square$

A natural question that arises is, what would be the correct analogue of the statement of the above theorem for general Blaschke products (and then for inner functions)? Where this analogy is unclear, one may consider an infinite cyclic group and ask whether there exists a meaningful connection between such a group and infinite Blaschke products (or inner functions). Some more concrete questions along these lines will be outlined at the end of the paper.

5. On nontrivial groups

In this section, we provide a complete solution to the question of the nontriviality of both $L(\mathcal{Q}_\theta)$ and $D(\mathcal{Q}_\theta)$. In a sense, the nontriviality of these spaces is closely related, as already observed in Proposition 2.2, where we have

$$D(\mathcal{Q}_\theta) \subseteq L(\mathcal{Q}_\theta).$$

We begin by addressing an equivalent formulation of the problem namely, determining when $D(\mathcal{Q}_\theta)$ is trivial. We proceed from simple cases to the most general ones. As we will see, the results for all cases, from particular to general, differ. Recall that given a set $\{\lambda_1, \ldots, \lambda_n\}$ of n distinct points in $\mathbb{D} \setminus \{0\}$, we define

$$\Lambda_n = \left\{ \frac{\lambda_1}{\lambda_1}, \frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1} \right\},$$

and regard it as a multiplicative group whenever we wish to view it as a group. The following result comes directly from existing results:

Proposition 5.1. *Let n be a prime number, $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{D} \setminus \{0\}$ be a set of n distinct scalars, and let $\{m_i\}_{i=1}^n \subseteq \mathbb{N}$. Define*

$$\theta = \prod_{i=1}^n b_{\lambda_i}^{m_i}.$$

Then $D(\mathcal{Q}_\theta) = \{z\}$ if and only if either not all the m_i's are equal or Λ_n is not a group.

Proof. Since n is prime, Corollary 4.5 implies that $D(\mathcal{Q}_\theta)$ is either $\{z\}$ or $\langle e^{\frac{2\pi i}{n}} z \rangle$. The equivalent formulation of the proposition now follows directly from Theorem 4.8. □

Note that $\Lambda_2 = \{\lambda_1/\lambda_1, \lambda_2/\lambda_1\}$ is a multiplicative group if and only if

$$\lambda_1 + \lambda_2 = 0.$$

Therefore, we have the following easy consequence: Let $\{\lambda_1, \lambda_2\}$ be a pair of distinct points in $\mathbb{D} \setminus \{0\}$, and let $m_1, m_2 \in \mathbb{N}$. Define

$$\theta = b_{\lambda_1}^{m_1} b_{\lambda_2}^{m_2}.$$

Then we have the following:

(1) $D(\mathcal{Q}_\theta) = \{z\}$ if and only if $m_1 \neq m_2$ or $\lambda_1 + \lambda_2 \neq 0$.
(2) $D(\mathcal{Q}_\theta) = \{z, -z\}$ if and only if $m_1 = m_2$ and $\lambda_1 + \lambda_2 = 0$.

Proposition 5.1 addresses the triviality of $D(\mathcal{Q}_\theta)$ in the case where $n = \#\mathcal{Z}(\theta)$ is a prime number. We now turn to the case of general natural numbers. Our eventual goal is to apply the prime factorization of n, and as a first step, we consider the case where n is a prime power:

Proposition 5.2. *Let θ be a finite Blaschke product. Assume that $\theta(0) \neq 0$ and $\#\mathcal{Z}(\theta) = p^k$ for some prime p and $k \in \mathbb{N}$. Then $D(\mathcal{Q}_\theta) = \{z\}$ if and only if there exists $\lambda \in \mathcal{Z}(\theta)$ such that*

$$mult_\theta(\lambda) \neq mult_\theta(e^{\frac{2\pi i}{p}}\lambda).$$

Proof. By Theorem 4.4, we know that $D(\mathcal{Q}_\theta)$ is a subgroup of $\langle e^{\frac{2\pi i}{p^k}} z\rangle$. Since every nontrivial subgroup of $\langle e^{\frac{2\pi i}{p^k}} z\rangle$ contains $e^{\frac{2\pi i}{p}} z$, it follows that $D(\mathcal{Q}_\theta) = \{z\}$ if and only if $e^{\frac{2\pi i}{p}} z \notin D(\mathcal{Q}_\theta)$. However, by Corollary 3.6, the latter occurs if and only if there exists $\lambda \in \mathcal{Z}(\theta)$ such that $mult_\theta(\lambda) \neq mult_\theta(e^{\frac{2\pi i}{p}}\lambda)$. □

We now consider the general case of $n = \#\mathcal{Z}(\theta)$:

Theorem 5.3. *Let θ be a finite Blaschke product. Assume that $\theta(0) \neq 0$. Consider the prime factorization of $n := \#\mathcal{Z}(\theta)$ as*

$$n = p_1^{k_1} \cdots p_m^{k_m}.$$

Then $D(\mathcal{Q}_\theta) = \{z\}$ if and only if for each $j \in \{1, \dots, m\}$, there exists $\lambda_j \in \mathcal{Z}(\theta)$ such that

$$mult_\theta(\lambda_j) \neq mult_\theta(e^{\frac{2\pi i}{p_j}}\lambda_j).$$

Proof. Suppose $D(\mathcal{Q}_\theta) = \{z\}$ and fix $j \in \{1, \dots, m\}$. Since $e^{\frac{2\pi i}{p_j}} z \notin D(\mathcal{Q}_\theta)$, part (3) of Corollary 3.6 implies

$$mult_\theta(\lambda_j) \neq mult_\theta(e^{\frac{2\pi i}{p_j}}\lambda_j),$$

for some $\lambda_j \in \mathcal{Z}(\theta)$. For the converse, we proceed by contradiction. Suppose, for the sake of argument, that $D(\mathcal{Q}_\theta) \neq \{z\}$. By Theorem 4.5, there exists a divisor $d(> 1)$ of n such that

$$D(\mathcal{Q}_\theta) = \langle e^{\frac{2\pi i}{d}} z\rangle.$$

Thus, there exists a prime factor p_j of n such that $p_j | d$; that is, $d = p_j \ell$ for some $\ell \in \mathbb{N}$. Hence

$$e^{\frac{2\pi i}{p_j}} z = \left(e^{\frac{2\pi i}{d}}\right)^\ell z \in \langle e^{\frac{2\pi i}{d}} z\rangle = D(\mathcal{Q}_\theta),$$

that is, $e^{\frac{2\pi i}{p_j}} z \in D(\mathcal{Q}_\theta)$ for some $j \in \{1, \dots, m\}$. By Theorem 3.5, it then follows that $mult_\theta(\lambda) = mult_\theta(e^{\frac{2\pi i}{p_j}}\lambda)$ for all $\lambda \in \mathcal{Z}(\theta)$, which leads to a contradiction. □

In the setting of the above result, we obtain the following by contrapositive which serves as an analog of the nontriviality question for $D(\mathcal{Q}_\theta)$ in the case of $L(\mathcal{Q}_\theta)$:

$$D(\mathcal{Q}_\theta) \neq \{z\},$$

if and only if there exist a prime factor p_j of n such that

$$mult_\theta(\lambda) = mult_\theta(e^{\frac{2\pi i}{p_j}}\lambda),$$

for all $\lambda \in \mathcal{Z}(\theta)$. Moreover, in this case, we have

$$\langle e^{\frac{2\pi i}{p_j}} z\rangle \subseteq D(\mathcal{Q}_\theta).$$

Theorem 5.4. *Let θ be a finite Blaschke product. Then $L(Q_\theta) = \{z\}$ if and only if the following conditions hold:*

(1) $\theta(0) \neq 0$.
(2) $\#\mathcal{Z}(\theta) \geq 2$.
(3) *For every non-constant affine map* $az + b$*, other than the identity, there exists* $\lambda \in \mathcal{Z}(\theta)$ *such that*

$$mult_\theta(\lambda) \neq mult_\theta\left(\frac{\bar{a}\lambda}{1 - \bar{b}\lambda}\right).$$

Proof. Suppose $L(Q_\theta) = \{z\}$. If $\theta(0) = 0$, then $L(Q_\theta) \supseteq \mathbb{C}$ (recall that here $\mathbb{C}$ refers to the set of all constant functions), which is uncountable and hence not possible. Therefore, $\theta(0) \neq 0$. If θ has only one zero, then by Proposition 3.1, $L(Q_\theta)$ is again uncountable. Hence, $\#\mathcal{Z}(\theta) \geq 2$. Part (c) follows from Lemma 3.3. The converse also follows easily from Lemma 3.3. □

In particular, we have: Let θ be a finite Blaschke product. Then $L(Q_\theta)$ is nontrivial if and only if any one of the following holds:

(1) $\theta(0) = 0$.
(2) There exist $n \in \mathbb{N}$ and a nonzero $\lambda \in \mathbb{D}$ such that $\theta = b_\lambda^n$.
(3) $\theta(0) \neq 0$, $\#\mathcal{Z}(\theta) \geq 2$, and there exists a non-constant affine map $az + b$ other than identity such that for all $\lambda \in \mathcal{Z}(\theta)$, we have

$$mult_\theta(\lambda) = mult_\theta\left(\frac{\bar{a}\lambda}{1 - \bar{b}\lambda}\right).$$

6. Blaschke products vanishing at 0

Here, we consider model spaces corresponding to finite Blaschke products that vanish at the origin. In this setting, the results often differ from those in the case where the Blaschke product does not vanish at the origin. We will also observe that the sets $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$ may contain constant functions. Keeping this in mind, throughout the paper, for any subset $X \subseteq \mathbb{C}$, when we write

$$X \subseteq D(\mathcal{Q}_\theta),$$

we mean that $D(\mathcal{Q}_\theta)$ contains constant functions that assume values in X. The same convention will be used for $L(\mathcal{Q}_\theta)$. For simplicity of notation, we write

$$L(\mathcal{Q}_\theta)^* = L(\mathcal{Q}_\theta) \setminus \mathbb{C} \text{ and } D(\mathcal{Q}_\theta)^* = D(\mathcal{Q}_\theta) \setminus \mathbb{D}.$$

The following is the $\alpha = 0$ case of [11, Theorem 3.1], along with its direct consequence:

Proposition 6.1. *Let* $\theta(z) = z^m$, $m \geq 2$. *Then*

(1) $L(\mathcal{Q}_\theta) = \{az + b : a, b \in \mathbb{C}, a \neq 0\} \cup \mathbb{C}$.
(2) $D(\mathcal{Q}_\theta) = \{az + b : a, b \in \mathbb{C}, a \neq 0, |a| + |b| \leq 1\} \cup \mathbb{D}$.
(3) $L(\mathcal{Q}_\theta)^* = \{az + b : a, b \in \mathbb{C}, a \neq 0\}$ *is a non-cyclic group.*
(4) $D(\mathcal{Q}_\theta)^*$ *is not a group.*

As for the proof of part (4) above, we simply note that $\frac{z}{2} \in D(\mathcal{Q}_\theta)^*$, but it is not invertible under composition.

Classifications of the elements of $D(\mathcal{Q}_\theta)$ for Blaschke products considered below were obtained in [9, Theorem 2.5]. Here, however, we present a different classification, which will be useful. Let $\text{Mob}(\mathbb{C}_\infty)$ denote the group of all Möbius transformations of $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Theorem 6.2. *Let $\lambda \in \mathbb{D}$ be a nonzero scalar, $n \in \mathbb{N}$, and let $\varphi \in \mathcal{S}(\mathbb{D})$ be a nonconstant function. Define*

$$\theta = zb_\lambda^n.$$

Then $C_\varphi \mathcal{Q}_\theta \subseteq \mathcal{Q}_\theta$ if and only if $\varphi \in \mathrm{Mob}(\mathbb{C}_\infty)$ and

$$\varphi\left(\frac{1}{\bar{\lambda}}\right) = \frac{1}{\bar{\lambda}}.$$

Proof. Suppose $\varphi \in D(\mathcal{Q}_\theta)$. As usual,

$$\frac{1}{1-\bar{\lambda}\varphi} = C_\varphi\left(\frac{1}{1-\bar{\lambda}z}\right) \in \mathcal{Q}_\theta,$$

implies that φ is a rational function. On the other hand, since (recall the basis functions from (2.2)) $\frac{1}{(1-\bar{\lambda}z)^n} \in \mathcal{Q}_\theta$, there exist scalars $\{c_0, c_1, \ldots, c_n\}$ such that

$$\frac{1}{(1-\bar{\lambda}z)^n} \circ \varphi = \frac{1}{(1-\bar{\lambda}\varphi)^n} = c_0 + \sum_{k=1}^{n} \frac{c_k}{(1-\bar{\lambda}z)^k}.$$

As φ is nonconstant, there exists $j \in \{1, \ldots, n\}$ such that $c_j \neq 0$. The right-hand side of the above equation has exactly one pole at $z = \frac{1}{\bar{\lambda}}$, of order at most n. This implies that $\dfrac{1}{1-\bar{\lambda}\varphi}$ has a simple pole at $z = \frac{1}{\bar{\lambda}}$. Therefore, there exist scalars a and $b(\neq 0)$ such that

$$\frac{1}{1-\bar{\lambda}\varphi} = a + \frac{b}{1-\bar{\lambda}z},$$

which implies that φ must be a Möbius transformation fixing the point $\dfrac{1}{\bar{\lambda}}$. For the converse direction, let $\varphi \in \mathrm{Mob}(\mathbb{C}_\infty)$ and assume that

$$\varphi\Big(\frac{1}{\bar{\lambda}}\Big) = \frac{1}{\bar{\lambda}}.$$

Then $1 - \bar{\lambda}\varphi(z)$ has a single zero at $\dfrac{1}{\bar{\lambda}}$. There exist scalars $\alpha, \beta \in \mathbb{C}$ such that

$$\frac{1}{1-\bar{\lambda}\varphi} = \frac{az+b}{1-\bar{\lambda}z} = \alpha + \frac{\beta}{1-\bar{\lambda}z}.$$

Thus

$$\frac{1}{(1-\bar{\lambda}\varphi)^k} \in span\left\{1, \frac{1}{1-\bar{\lambda}z}, \ldots, \frac{1}{(1-\bar{\lambda}z)^k}\right\} \subseteq \mathcal{Q}_\theta,$$

for all $k = 1, \ldots, n$, and consequently $C_\varphi(\mathcal{Q}_\theta) \subseteq \mathcal{Q}_\theta$. This completes the proof of the theorem. □

Analogous result for $L(\mathcal{Q}_\theta)$ is also true with the same lines of proof. Thus, we give representations of $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$ as follows:

Corollary 6.3. *Let $\lambda \in \mathbb{D} \setminus \{0\}$ and let $n \in \mathbb{N}$. Define*

$$\theta = zb_\lambda^n.$$

Then

$$L(\mathcal{Q}_\theta) = \left\{\varphi \in \mathrm{Mob}(\mathbb{C}_\infty) : \varphi\left(\frac{1}{\bar{\lambda}}\right) = \frac{1}{\bar{\lambda}}\right\} \cup \mathbb{C},$$

and

$$D(\mathcal{Q}_\theta) = \left\{\varphi \in \mathcal{S}(\mathbb{D}) \cap \mathrm{Mob}(\mathbb{C}_\infty) : \varphi\left(\frac{1}{\bar{\bar{\lambda}}}\right) = \frac{1}{\bar{\bar{\lambda}}}\right\} \cup \mathbb{D}.$$

Moreover, in this case, $D(\mathcal{Q}_\theta)^* \cap \mathrm{Aut}(\mathbb{D})$ *is uncountable, and* $L(\mathcal{Q}_\theta)^*$ *is an infinite group that is not cyclic.*

Proof. The first two identities follow from the previous theorem. Pick $\varphi \in D(\mathcal{Q}_\theta)^* \cap \mathrm{Aut}(\mathbb{D})$. In particular, there exist $\alpha \in \mathbb{R}$ and $a \in \mathbb{D}$ such that

$$\varphi(z) = e^{i\alpha} b_a.$$

The condition $\varphi\left(\frac{1}{\bar{\bar{\lambda}}}\right) = \frac{1}{\bar{\bar{\lambda}}}$ is equivalent to the identity

$$e^{i\alpha}\bar{\lambda}(1 - a\bar{\lambda}) = \bar{\lambda} - \bar{a}.$$

From this, we deduce that $|\lambda| = \left|\frac{a-\lambda}{1-a\bar{\lambda}}\right|$, which implies

$$1 - |\lambda|^2 = 1 - \left|\frac{a-\lambda}{1-a\bar{\lambda}}\right|^2.$$

Since

$$1 - \left|\frac{a-\lambda}{1-a\bar{\lambda}}\right|^2 = \frac{(1-|\lambda|^2)(1-|a|^2)}{|1-a\bar{\lambda}|^2},$$

and $1 - |\lambda|^2 \neq 0$, the above identity simplifies to

$$|1 - a\bar{\lambda}|^2 + |a|^2 = 1,$$

equivalently,

$$(1 + |\lambda|^2)|a|^2 - 2\mathrm{Re}(a\bar{\lambda}) = 0.$$

This implies that a lies on the circle $C(\lambda)$, centered at $\frac{\lambda}{1+|\lambda|^2}$ with radius $\frac{|\lambda|}{1+|\lambda|^2}$, that is, $a \in C(\lambda)$, where

$$C(\lambda) = \left\{z \in \mathbb{C} : \left|z - \frac{\lambda}{1+|\lambda|^2}\right|^2 = \left(\frac{|\lambda|}{1+|\lambda|^2}\right)^2\right\}.$$

Since $0 < |\lambda| < 1$, it follows that

$$\frac{|\lambda|}{1+|\lambda|^2} < \frac{1}{2},$$

and hence

$$C(\lambda) \subseteq \mathbb{D}.$$

In particular, the set $D(\mathcal{Q}_\theta)^* \cap \mathrm{Aut}(\mathbb{D})$ is uncountable. Since $D(\mathcal{Q}_\theta)^* \subseteq L(\mathcal{Q}_\theta)^*$, it follows that $L(\mathcal{Q}_\theta)^*$ is also uncountable. If a Möbius map φ fixes $\frac{1}{\bar{\lambda}}$, then its inverse φ^{-1} also fixes $\frac{1}{\bar{\lambda}}$. Hence $\varphi^{-1} \in L(\mathcal{Q}_\theta)^*$, which implies that $L(\mathcal{Q}_\theta)^*$ forms a group with uncountably many elements. Therefore, it cannot be a cyclic group. This concludes the proof. □

The proof of the above corollary implies that

$$D(\mathcal{Q}_\theta)^* \cap \operatorname{Aut}(\mathbb{D}) = \left\{ \frac{1}{\bar{\lambda}} \overline{b_a(\lambda)} b_a : a \in \gamma \right\}.$$

Note also that $D(\mathcal{Q}_\theta)^*$ is a group if and only if

$$D(\mathcal{Q}_\theta)^* \cap \operatorname{Aut}(\mathbb{D}) = D(\mathcal{Q}_\theta)^*.$$

Next, we proceed to the case of Blaschke products that have the origin as a zero of higher multiplicity, along with one nonzero zero. An incomplete statement addressing this setting appeared in [9, Theorem 2.9]. In particular, it omitted the most nontrivial aspect of the result. Our formulation corrects and completes that statement, and we also provide a full proof.

Theorem 6.4. *Let $m \geq 2$, $n \geq 1$, and let $\lambda \in \mathbb{D} \setminus \{0\}$. Define*

$$\theta = z^m b_\lambda^n.$$

Then

$$L(\mathcal{Q}_\theta) = \left\{ (1 - \bar{\lambda} a) z + a : a \neq \frac{1}{\bar{\lambda}} \right\} \cup \left(\mathbb{C} \setminus \left\{ \frac{1}{\bar{\lambda}} \right\} \right),$$

and

$$D(\mathcal{Q}_\theta) = \{z\} \cup \mathbb{D}.$$

Proof. In view of (2.1) and (2.2), we have

$$\mathcal{Q}_\theta = \operatorname{span} \left\{ 1, z, \ldots, z^{m-1}, \frac{1}{1 - \bar{\lambda} z}, \ldots, \frac{1}{\left(1 - \bar{\lambda} z\right)^n} \right\}.$$

If $\varphi \in \mathcal{S}(\mathbb{D})$ is constant, then C_φ maps $H^2(\mathbb{D})$ into the space of constant functions. Therefore, only subspaces that contain constants can be invariant under C_φ. On the other hand, If $\varphi(z) = z$, then the operator C_φ is the identity operator, implying that $\mathcal{Q}_\theta$ is invariant under C_φ. Moreover, a simple computation shows that $L(\mathcal{Q}_\theta)$ contains the functions on the right side of the identity in the statement.
For the converse direction, assume that $\mathcal{Q}_\theta$ is invariant under C_φ. In particular,

$$C_\varphi \left(\frac{1}{1 - \bar{\lambda} z} \right) \in \mathcal{Q}_\theta,$$

and hence there exist scalars $c_1, \ldots, c_{m-1}$ and $d_1, \ldots, d_n$, not all zero, such that

$$\frac{1}{1 - \bar{\lambda} \varphi} = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + \sum_{k=1}^{n} \frac{d_k}{(1 - \bar{\lambda} z)^k}. \tag{6.1}$$

In particular, φ is a rational function of the form $\varphi = \frac{p}{q}$, where p and q are polynomials with no common factors. We can then rewrite (6.1) as follows:

$$\frac{q}{q - \bar{\lambda} p} = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + \sum_{k=1}^{n} \frac{d_k}{(1 - \bar{\lambda} z)^k}. \tag{6.2}$$

If $d_j = 0$ for all $j \in \{1, \ldots, n\}$, then this contradicts the assumption that p and q have no common factors. Therefore, we may assume that $d_j \neq 0$ for some $j \in \{1, \ldots, n\}$. As

$$C_\varphi\Big(\frac{1}{(1-\bar{\lambda}z)^n}\Big) = \frac{1}{(1-\bar{\lambda}\varphi)^n} \in \mathcal{Q}_\theta,$$

there exist scalars $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ and $\beta_1, \ldots, \beta_n$ such that

$$\frac{1}{(1-\bar{\lambda}\varphi)^n} = \alpha_0 + \alpha_1 z + \cdots + \alpha_{m-1} z^{m-1} + \sum_{k=1}^{n} \frac{\beta_k}{(1-\bar{\lambda}z)^k}$$

Since the left-hand side of the identity has a pole $\frac{1}{\bar{\lambda}}$ of order at most n, it follows that $\dfrac{1}{1-\bar{\lambda}\varphi}$ must have a simple pole at $\frac{1}{\bar{\lambda}}$. Then (6.1) becomes

$$\frac{1}{1-\bar{\lambda}\varphi} = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + \frac{d_1}{1-\bar{\lambda}z},$$

where $d_1 \neq 0$. Since the right-hand side of the above identity has a simple pole at $\frac{1}{\bar{\lambda}}$, it follows that $1 - \bar{\lambda}\varphi$ must vanish at $\frac{1}{\bar{\lambda}}$, that is,

$$\varphi\left(\frac{1}{\bar{\lambda}}\right) = \frac{1}{\bar{\lambda}}.$$

As $z \in \mathcal{Q}_\theta$, we have $\varphi \in \mathcal{Q}_\theta$. On the other hand, $\varphi(\frac{1}{\bar{\lambda}}) \neq \infty$ implies that φ is a polynomial of degree at most $m-1$. It follows that

$$\varphi(z) = b_0 + b_1 z + \cdots + b_{m-1} z^{m-1},$$

for some constants $b_0, b_1, \ldots, b_{m-1}$. If $\deg \varphi \geq 2$, then $1 - \bar{\lambda}\varphi$ is also polynomial of degree at least 2. However, this is not possible, since $\dfrac{1}{1-\bar{\lambda}\varphi}$ has only a simple pole. Thus, we must have

$$\varphi(z) = b_0 + b_1 z,$$

with $b_1 \neq 0$, because φ is nonconstant. Consequently, $1 - \bar{\lambda}\varphi$ is polynomial of degree 1 and vanishing at $\frac{1}{\bar{\lambda}}$. There exists a nonzero scalar γ such that

$$1 - \bar{\lambda}\varphi(z) = \gamma(1 - \bar{\lambda}z).$$

This implies

$$\frac{1}{1-\bar{\lambda}\varphi(z)} = \frac{\frac{1}{\gamma}}{1-\bar{\lambda}z},$$

Cross-multiplying the above identity yields the desired representation of φ, which now belongs to $L(\mathcal{Q}_\theta)$. For $D(\mathcal{Q}_\theta)$, Lemma 2.3 implies that $\varphi(z) = z$ for all $z \in \mathbb{D}$. This completes the proof. □

The following is now immediate:

Corollary 6.5. *Let $m \geq 2$ and $n \geq 1$ be natural numbers, and let $\lambda \in \mathbb{D} \setminus \{0\}$. Define*

$$\theta = z^m b_\lambda^n.$$

Then $D(\mathcal{Q}_\theta)^*$ is a trivial group and $L(\mathcal{Q}_\theta)^*$ is an uncountable group.

With this result, we now have a clearer picture of the structure of the sets $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$ when $\theta = z^m b_\lambda^n$ for $\lambda \in \mathbb{D} \setminus \{0\}$ and $m, n \geq 1$.

7. Möbius transformations

As in the previous section, we continue here under the assumption that the finite Blaschke product vanishes at the origin. Here, we assume that the finite Blaschke product has at least two distinct zeros, aside from the origin. The main difference in this setting is that the sets $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$ will contain more Möbius transformations. We first make a remark about the group structure of $L(\mathcal{Q}_\theta)$ in the case where the Blaschke product vanishes at the origin.

Remark 7.1. *Consider a finite Blaschke product θ with $\theta(0) = 0$. In this case, the constant functions belong to $L(\mathcal{Q}_\theta)$, but these are not invertible under composition. Hence, $L(\mathcal{Q}_\theta)$ cannot be a group under composition.*

Given a Möbius transformation $\varphi(z) = \dfrac{az+b}{cz+d}$, we construct another Möbius transformation $\tilde{\varphi}$ as

$$\tilde{\varphi}(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}.$$

Moreover, given a self-map f and a natural number n, we write

$$f^{[n]} = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}.$$

These transformations behave naturally:

Lemma 7.2. *Let φ be a Möbius transformation, and let $n \in \mathbb{N}$. We have the following:*
- (i) *$\varphi \in \mathcal{S}(\mathbb{D})$ if and only if $\tilde{\varphi} \in \mathcal{S}(\mathbb{D})$.*
- (ii) *$\varphi^{[n]} = z$ if and only if $\tilde{\varphi}^{[n]} = z$.*

Proof. Define $\eta(z) = \overline{\varphi(\bar{z})} = \dfrac{\bar{a}z + \bar{b}}{\bar{c}z + \bar{d}}$. Then

$$\eta = \bar{z} \circ \varphi \circ \bar{z}.$$

Given that $\eta^{-1}(z) = \dfrac{\bar{d}z - \bar{b}}{-\bar{c}z + \bar{a}}$, we have

$$\tilde{\varphi} = \frac{1}{z} \circ \eta^{-1} \circ \frac{1}{z}.$$

In view of this, we see that φ is a self-map of $\mathbb{D}$ if and only if η is a self-map of $\mathbb{D}$ if and only if η^{-1} is a self-map of

$$\{z : |z| > 1\} \cup \{\infty\},$$

if and only if $\tilde{\varphi}$ is a self-map of $\mathbb{D}$. This completes the proof of part (i). For (ii), we proceed similarly:

$$\varphi^{[n]} = z \Leftrightarrow \eta^{[n]} = z \Leftrightarrow (\eta^{-1})^{[n]} = z \Leftrightarrow \tilde{\varphi}^{[n]} = z.$$

This completes the proof of the lemma. □

In the following, we consider finite Blaschke products θ satisfying $\theta(0)=0$, $\theta'(0)\neq 0$, and $\#(\mathcal{Z}(\theta)\setminus\{0\})\geq 2$. These conditions imply that there exists a finite Blaschke product $\hat{\theta}$, not vanishing at the origin, such that $\theta = z\hat{\theta}$ and $\#(\mathcal{Z}(\hat{\theta}))\geq 2$. The following is a part of [9, Lemma 2.7].

Lemma 7.3. *Let θ be a finite Blaschke product satisfying $\theta(0)=0$, $\theta'(0)\neq 0$, and $\#(\mathcal{Z}(\theta)\setminus\{0\})\geq 2$. Let φ be a non-constant holomorphic self-map of $\mathbb{C}_\infty$. If $\varphi\in L(\mathcal{Q}_\theta)$, then φ is a Möbius transformation of the form*

$$\varphi(z)=\frac{az+b}{cz+d},$$

for some complex numbers a,b,c,d satisfying $ad-bc\neq 0$.

Note that the full-length version of Lemma 2.7 in [9] concerns representations of functions $\varphi\in L(\mathcal{Q}_\theta)$ for the class of finite Blaschke products θ described above. We now use the above part of the result to provide an alternative classification of functions in the sets $D(\mathcal{Q}_\theta)$ and $L(\mathcal{Q}_\theta)$:

Theorem 7.4. *Let θ be a finite Blaschke product satisfying $\theta(0)=0$, $\theta'(0)\neq 0$, and let $\#(\mathcal{Z}(\theta)\setminus\{0\})\geq 2$. Then $\varphi\in L(\mathcal{Q}_\theta)$ if and only if φ is a Möbius transformation and*

$$mult_\theta(\lambda)=mult_\theta(\tilde{\varphi}(\lambda)),$$

for all $\lambda\in\mathcal{Z}(\theta)\setminus\{0\}$. More over, we have

$$D(\mathcal{Q}_\theta)=L(\mathcal{Q}_\theta)\cap\mathcal{S}(\mathbb{D}).$$

Proof. Suppose $\varphi\in L(\mathcal{Q}_\theta)$. By Lemma 7.3, φ is a Möbius transformation. Let

$$\varphi(z)=\frac{az+b}{cz+d}.$$

Fix $\lambda\in\mathcal{Z}(\theta)\setminus\{0\}$, and let $m=mult_\theta(\lambda)$. Now

$$\frac{1}{1-\bar{\lambda}z}\circ\varphi=\frac{1}{1-\bar{\lambda}\varphi}=\frac{1}{1-\bar{\lambda}\dfrac{az+b}{cz+d}}=\frac{\alpha z+\beta}{1-\overline{\dfrac{\overline{a}\lambda-\overline{c}}{-\overline{b}\lambda+\overline{d}}}\cdot z}\in\mathcal{Q}_\theta,$$

for some scalars α and β. Then

$$\frac{1}{1-\bar{\lambda}\varphi}=\frac{\alpha z+\beta}{1-\overline{\tilde{\varphi}(\lambda)}z}=\gamma+\frac{\delta}{1-\overline{\tilde{\varphi}(\lambda)}z}\in\mathcal{Q}_\theta, \tag{7.1}$$

for some scalars $\delta\neq 0,\gamma\in\mathbb{C}$. This shows that $\tilde{\varphi}(\lambda)\in\mathcal{Z}(\theta)$. Moreover, for each $k\in\{1,\ldots,m\}$, we have

$$\frac{1}{(1-\bar{\lambda}\varphi)^k}=\frac{1}{(1-\bar{\lambda}z)^k}\circ\varphi\in\mathcal{Q}_\theta,$$

and hence

$$\left(\frac{\delta}{1-\overline{\tilde{\varphi}(\lambda)}z}\right)^k=\left(\frac{1}{1-\bar{\lambda}\varphi}-\gamma\right)^k=\sum_{j=0}^{k}\frac{c_j}{(1-\bar{\lambda}z)^j}\in\mathcal{Q}_\theta$$

for some scalars $c_1,\ldots,c_k$. Therefore, $\tilde{\varphi}(\lambda)\in\mathcal{Z}(\theta)$ and

$$m=mult_\theta(\lambda)\leq mult_\theta(\tilde{\varphi}(\lambda)).$$

If we set $\tilde{\varphi}^{[0]} := z$, then, by induction, we have

$$mult_\theta(\tilde{\varphi}^{[k]}(\lambda)) \leq mult_\theta(\tilde{\varphi}^{[k+1]}(\lambda)),$$

for all $k \in \mathbb{Z}_+$. This implies

$$\tilde{\varphi}^{[k+1]}(\lambda) \in \mathcal{Z}(\theta),$$

for all $k \in \mathbb{N}$. Since θ has only finitely many zeros, it follows that $\tilde{\varphi}^{[k]}(\lambda) = \lambda$ for some $k \in \mathbb{N}$. Thus, by (7.1), we have

$$mult_\theta(\lambda) = mult_\theta(\tilde{\varphi}(\lambda)).$$

Conversely, suppose φ is Möbius transformation of the form $\varphi(z) = \dfrac{az+b}{cz+d}$ with $ad-bc \neq 0$, and assume that $mult_\theta(\lambda) = mult_\theta(\tilde{\varphi}(\lambda))$ for all $\lambda \in \mathcal{Z}(\theta) \setminus \{0\}$. Fix $\lambda \in \mathcal{Z}(\theta) \setminus \{0\}$, and let $m = mult_\theta(\lambda)$. As before,

$$\frac{1}{1-\bar{\lambda}\varphi} = \gamma + \frac{\delta}{1-\overline{\tilde{\varphi}(\lambda)}z},$$

for some scalars $\delta \neq 0$ and $\gamma \in \mathbb{C}$. Since $mult_\theta(\tilde{\varphi}(\lambda)) = m$, we have

$$\frac{1}{(1-\overline{\tilde{\varphi}(\lambda)}z)^j} \in \mathcal{Q}_\theta,$$

for all $j = 1, \ldots, m$. As $1 \in \mathcal{Q}_\theta$, we also have

$$\frac{1}{(1-\bar{\lambda}z)^k} \circ \varphi = \frac{1}{(1-\bar{\lambda}\varphi)^k} = \left(\gamma + \frac{\delta}{1-\overline{\tilde{\varphi}(\lambda)}z}\right)^k \in \mathcal{Q}_\theta,$$

for all $k \in \{1, \ldots, m\}$. Thus, C_φ maps every basis element of $\mathcal{Q}_\theta$ into $\mathcal{Q}_\theta$, and hence $\varphi \in L(\mathcal{Q}_\theta)$. The identity $D(\mathcal{Q}_\theta) = L(\mathcal{Q}_\theta) \cap \mathcal{S}(\mathbb{D})$ follows trivially. This completes the proof. $\square$

The similar proof as in Theorem 7.4 applies to the following result. This leads to yet another characterization, as noted in [9, Lemma 2.10]: Let θ be a finite Blaschke product. Suppose $\theta(0) = 0$ with multiplicity at least 2 and $\#(\mathcal{Z}(\theta) \setminus \{0\}) \geq 2$. Then $\varphi \in L(\mathcal{Q}_\theta)$ if and only if φ is a Möbius transformation of the form

$$\varphi(z) = \frac{az+b}{cz+d}$$

for some scalars a, b, c, d satisfying $ad - bc \neq 0$, and

$$mult_\theta(\lambda) = mult_\theta(\tilde{\varphi}(\lambda)),$$

for all $\lambda \in \mathcal{Z}(\theta) \setminus \{0\}$ and

$$mult_\theta(-c/d) \geq mult_\theta(0) - 1.$$

Further, $D(\mathcal{Q}_\theta) = L(\mathcal{Q}_\theta) \cap \mathcal{S}(\mathbb{D})$.

As an application, we have:

Corollary 7.5. *Let θ be a finite Blaschke product with $\theta(0) = 0$ and $\#(\mathcal{Z}(\theta) \setminus \{0\}) \geq 2$, and let $\varphi \in D(Q_\theta)$ be a non-constant function. Then:*

(1) $\varphi \in \mathrm{Aut}(\mathbb{D})$.

(2) There exists $m \in \mathbb{N}$ such that $\varphi^{[m]} = z$.

Proof. Fix $\lambda \in A := \mathcal{Z}(\theta) \setminus \{0\}$. By (7.1), there exist scalars $\delta \neq 0$ and $\gamma \in \mathbb{C}$ such that

$$\frac{1}{1-\bar{\lambda}\varphi} = \gamma + \frac{\delta}{1-\overline{\tilde{\varphi}(\lambda)}z}$$

and $\tilde{\varphi}(\lambda) \in \mathcal{Z}(\theta)$. If $\tilde{\varphi}(\lambda) = 0$, then this would imply φ is constant, which is not possible. Thus, $\tilde{\varphi}(\lambda) \in A$. Since $\tilde{\varphi}$ is Möbius (and in particular, injective) map, it defines a permutation on A. That is, $\tilde{\varphi} \in S_N$, where

$$N := \#A.$$

There exist $m \in \mathbb{N}$ (for instance, $m = N!$) such that

$$\tilde{\varphi}^{[m]} = z \text{ on } A.$$

That is, every point of A is a fixed point of $\tilde{\varphi}$. As a consequence of the Schwarz lemma, we conclude that

$$\tilde{\varphi}^{[m]} = z \text{ on } \mathbb{D},$$

since a holomorphic self-map of $\mathbb{D}$ with at least two fixed points must be the identity. By applying part (ii) of Proposition 7.2, we obtain that $\varphi^{[m]} = z$. Thus,

$$\varphi \circ \varphi^{[m-1]} = \varphi^{[m-1]} \circ \varphi = z.$$

That is, $\varphi : \mathbb{D} \to \mathbb{D}$ is bijective. Hence φ is an automorphism of $\mathbb{D}$. □

In particular, in the above setting, φ is either a rational elliptic automorphism of $\mathbb{D}$ or $\varphi = z$. We also have the following result as a consequence.

Corollary 7.6. *Let θ be a finite Blaschke product. Suppose $\theta(0) = 0$ and $\#(\mathcal{Z}(\theta) \setminus \{0\}) \geq 2$. Then $D(\mathcal{Q}_\theta)^*$ forms a group.*

Proof. Assume that $\varphi \in D(\mathcal{Q}_\theta)^*$. This implies $\varphi^{[k]} \in D(\mathcal{Q}_\theta)^*$ for all $k \in \mathbb{N}$. By Corollary 7.5, there exists $n \in \mathbb{N}$ such that

$$\varphi^{-1} = \varphi^{[n-1]} \in D(\mathcal{Q}_\theta)^*,$$

which implies that $D(\mathcal{Q}_\theta)^*$ is a group. □

As for the group structure of $L(\mathcal{Q}_\theta)^*$, we have the following:

Corollary 7.7. *Let θ be a finite Blaschke product. Suppose $\theta(0) = 0$ and $\#(\mathcal{Z}(\theta) \setminus \{0\}) \geq 3$. Then $L(\mathcal{Q}_\theta)^*$ forms a group.*

Proof. Assume that $\varphi \in L(\mathcal{Q}_\theta)^*$. Set $A := \mathcal{Z}(\theta) \setminus \{0\}$. Then as in the proof of Corollary 7.5, there exist $m \in \mathbb{N}$ such that

$$\tilde{\varphi}^{[m]} = z,$$

on A. Therefore the Möbius map $\tilde{\varphi}^{[m]}$ has at least $\#A (\geq 3)$ fixed points. This implies

$$\tilde{\varphi}^{[m]} = z \text{ on } \mathbb{C}_\infty.$$

Again, by reasoning similar to that used in the above corollary, we obtain that $\varphi^{-1} \in L(\mathcal{Q}_\theta)^*$, which implies that $L(\mathcal{Q}_\theta)^*$ is a group. □

Suppose $\#A = 2$. In this case, $\tilde{\varphi}^{[m]}$ has two fixed points in $\mathbb{D}$ for some $m \in \mathbb{N}$. It is not known whether $L(\mathcal{Q}_\theta)^*$ forms a group in this case.

8. An example

The focus of this section is an example that was presented as an illustration of Theorem 2.4 in [9], and which appeared immediately after Corollary 2.4 in the same reference. We point out that the conclusion drawn in that example is not necessarily valid in all cases. In this section, we also correct the error and provide a precise statement that accurately reflects the situation illustrated by the example in [9].

We recall the setting of [9]: Fix distinct points $\{\lambda_1, \ldots, \lambda_s\}$ from $\mathbb{D} \setminus \{0\}$ and fix a natural number $n(\geq 2)$. Set

$$a = e^{\frac{2\pi i}{n}}.$$

Define

$$\theta = \prod_{j=1}^{s} \left(\prod_{k=1}^{n} b_{a^k \lambda_j} \right)^{m_j}.$$

In this case, the statement following Corollary 2.4 in [9] asserts that

$$D(\mathcal{Q}_\theta) = \{z, az, \ldots, a^{n-1}z\}.$$

We first point out that this conclusion is not correct: Consider the case with $n = 2$, and

$$a = -1.$$

We choose

$$s = 2, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{i}{2},$$

with $m_1 = m_2 = 1$. Then

$$\theta = b_{-\frac{1}{2}} b_{\frac{1}{2}} b_{-\frac{i}{2}} b_{\frac{i}{2}},$$

and according to [9], one concludes that

$$D(\mathcal{Q}_\theta) = \{z, -z\}. \tag{8.1}$$

We apply the results proved in this paper to show that this conclusion is incorrect. Note that

$$\mathcal{Z}(\theta) = \left\{ \pm\frac{1}{2}, \pm\frac{i}{2} \right\},$$

and all zeros have multiplicity one. In particular, we have

$$mult_\theta(\lambda) = mult_\theta(i\lambda),$$

for all $\lambda \in \mathcal{Z}(\theta)$. By part (2) of Corollary 3.6, we have $iz \in D(\mathcal{Q}_\theta)$, and hence

$$\langle iz \rangle = \{\pm z, \pm iz\} \subseteq D(\mathcal{Q}_\theta).$$

On the other hand, by Lemma 4.4, we have $D(\mathcal{Q}_\theta) \subseteq \langle iz \rangle$, and consequently

$$D(\mathcal{Q}_\theta) = \{\pm z, \pm iz\}.$$

This is clearly different from the identity claimed in (8.1). In this case, observe that all the zeros are simple. Hence, even if one assumes that the zeros of θ are all distinct, claim (8.1) is false. However, there is a fix to this claim, which we state as follows:

Theorem 8.1. *Let $n \geq 2$ be a natural number, and let $\{\lambda_1, \ldots, \lambda_s\} \subseteq \mathbb{D} \setminus \{0\}$. Set*

$$\theta = \prod_{j=1}^{s} \left(\prod_{k=1}^{n} b_{a^k \lambda_j} \right)^{m_j},$$

where

$$a = e^{\frac{2\pi i}{n}}.$$

Assume that $\{|\lambda_1|, \ldots, |\lambda_s|\}$ is a set of distinct real numbers. Then

$$D(\mathcal{Q}_\theta) = \langle az \rangle.$$

Proof. By definition of θ, we have

$$\mathcal{Z}(\theta) = \bigcup_{j=1}^{s} \left\{ \lambda_j, a\lambda_j, a^2\lambda_j, \ldots, a^{n-1}\lambda_j \right\},$$

with the property that

$$mult_\theta(a^k \lambda_j) = m_j,$$

for all $k = 1, \ldots, n$. Therefore, $mult_\theta(\lambda) = mult_\theta(a\lambda)$ for all $\lambda \in \mathcal{Z}(\theta)$. Thus, by part (2) of Corollary 3.6, this implies that $az \in D(\mathcal{Q}_\theta)$, and hence

$$\langle az \rangle \subseteq D(\mathcal{Q}_\theta),$$

as $D(\mathcal{Q}_\theta)$ is a group. To prove the reverse inclusion, we pick $\varphi \in D(\mathcal{Q}_\theta)$. We know by Theorem 4.1 that

$$\varphi = \omega z,$$

for some $\omega \in \mathbb{T}$. Since $\lambda_1 \in \mathcal{Z}(\theta)$, again, by part (2) of Corollary 3.6, $\omega\lambda_1 \in \mathcal{Z}(\theta)$ and $|\omega\lambda_1| = |\lambda_1|$. As $|\lambda_p| \neq |\lambda_q|$ for all $p \neq q$, it follows that

$$\omega\lambda_1 \in \left\{ \lambda_1, a\lambda_1, \ldots, a^{n-1}\lambda_1 \right\},$$

and hence, there exists $k \in \{0, 1, \ldots, n-1\}$ such that

$$\omega = a^k.$$

This proves that $D(\mathcal{Q}_\theta) \subseteq \langle az \rangle$, and consequently, we have $D(\mathcal{Q}_\theta) = \langle az \rangle$. □

The example at the beginning of this section shows that the additional assumption $|\lambda_i| \neq |\lambda_j|$ for all $i \neq j$ (as opposed to $\lambda_i \neq \lambda_j$ for all $i \neq j$) cannot be omitted from the theorem. There are other ways to obtain a similar conclusion. For instance, we have the following: Let $n \geq 2$ be a natural number, and let $\{\lambda_1, \ldots, \lambda_s\} \subseteq \mathbb{D} \setminus \{0\}$ be a set of distinct scalars. Set

$$\theta = \prod_{j=1}^{s} \left(\prod_{k=1}^{n} b_{a^k \lambda_j} \right)^{m_j},$$

where $a = e^{\frac{2\pi i}{n}}$. Assume that $m_i \neq m_j$ for all $i \neq j$. Then

$$D(\mathcal{Q}_\theta) = \langle az \rangle.$$

The proof of $\langle az \rangle \subseteq D(\mathcal{Q}_\theta)$ is the same. As above, pick $\varphi = \omega z \in D(\mathcal{Q}_\theta)$ for some $\omega \in \mathbb{T}$. As $\lambda_1 \in \mathcal{Z}(\theta)$ and $\omega z \in D(\mathcal{Q}_\theta)$, it follows from part (2) of Corollary 3.6 that $\omega\lambda_1 \in \{\lambda_1, a\lambda_1, \ldots, a^{n-1}\lambda_1\}$. The remainder of the proof proceeds as before.

Once again, our example shows that the additional condition on multiplicities cannot, in general, be omitted unless the $|\lambda_j|$'s are all distinct.

9. Concluding remarks

In this concluding section, we offer some additional remarks on finite-dimensional model spaces that are invariant under composition operators. The following result, while elementary, is quite intriguing. Moreover, the same conclusion holds in the context of the space $L(\mathcal{Q}_\theta)$ (that is, for C_φ with $\varphi \in L(\mathcal{Q}_\theta)$):

Theorem 9.1. *Let θ be a finite Blaschke product that does not vanish at the origin, and let $\varphi \in \mathcal{S}(\mathbb{D})$. Then*

$$C_\varphi(\mathcal{Q}_\theta) \subseteq \mathcal{Q}_\theta,$$

if and only if

$$C_\varphi(\mathcal{Q}_\theta) = \mathcal{Q}_\theta.$$

Proof. Suppose $C_\varphi(\mathcal{Q}_\theta) \subseteq \mathcal{Q}_\theta$. By Theorem 3.5, we know that φ is a rotation. Pick f and g from $\mathcal{Q}_\theta$ so that $C_\varphi f = C_\varphi g$, that is,

$$f \circ \varphi = g \circ \varphi.$$

As φ is rotation, it follows that

$$f \equiv g,$$

and hence C_φ is injective. Since $\mathcal{Q}_\theta$ is a finite-dimensional space, the rank–nullity theorem implies that C_φ is surjective. □

The situation changes when we assume that $\theta(0) = 0$: For $\varphi \in \mathcal{S}(\mathbb{D})$, assume that $C_\varphi(\mathcal{Q}_\theta) \subseteq \mathcal{Q}_\theta$. Then

$$C_\varphi(\mathcal{Q}_\theta) = \mathcal{Q}_\theta,$$

if and only if

$$\theta = z.$$

Indeed, if $\theta = z$, then $\mathcal{Q}_\theta$ is one dimensional, and $\mathcal{Q}_\theta = \mathbb{C}$. Then

$$1 \circ \varphi = 1,$$

for all $\varphi \in \mathcal{S}(\mathbb{D})$, and hence $C_\varphi(\mathcal{Q}_\theta) = \mathcal{Q}_\theta$. For the converse direction, assume that $\theta \neq z$. Given that $\theta(0) = 0$, there exists a non-constant finite Blaschke product B such that $\theta = zB$, which implies that

$$\mathbb{C} \subsetneqq \mathcal{Q}_\theta.$$

For a constant map $\varphi \in \mathcal{S}(\mathbb{D})$, as

$$C_\varphi(\mathcal{Q}_\theta) = \mathbb{C} \subsetneq \mathcal{Q}_\theta,$$

we conclude that $C_\varphi(\mathcal{Q}_\theta) \neq \mathcal{Q}_\theta$. This proves that $\theta = z$ implies $C_\varphi(\mathcal{Q}_\theta) = \mathcal{Q}_\theta$.

In particular, for $\theta(0) = 0$, we have the following:

(1) If $\theta = z$ on $\mathbb{D}$, then $C_\varphi(\mathcal{Q}_\theta) = \mathcal{Q}_\theta$ for all $\varphi \in \mathcal{S}(\mathbb{D})$.
(2) If $\theta \neq z$, then there exists $\varphi \in \mathcal{S}(\mathbb{D})$ such that $C_\varphi(\mathcal{Q}_\theta) \subseteq \mathcal{Q}_\theta$ and $C_\varphi(\mathcal{Q}_\theta) \neq \mathcal{Q}_\theta$.

In [8] (see also [1]), it was pointed out that, given a composition operator, there always exists a shift-invariant subspace that is invariant under the composition operator. A similar question can be posed for model spaces (that is, backward shift-invariant subspaces). The following result addresses this issue explicitly.

Remark 9.2. *If $\theta = z$, then $\mathcal{Q}_\theta = \mathbb{C}$ is trivially invariant under every composition operator. Next, consider an analytic self-map φ of $\mathbb{D}$ that is not a Möbius map (for instance, $\varphi = z^2$). Since φ is neither a rotation nor a Möbius map, it follows from the results of this paper that $\varphi \notin D(\mathcal{Q}_\theta)$ for all θ other than z. That is, $\mathcal{Q}_\theta$ is not invariant under C_φ for any θ other than z. Hence, there exist many composition operators for which no non-trivial model space is invariant. That is, except for $\mathbb{C}$, all other model spaces fail to be invariant under any composition operator induced by a non-Möbius maps.*

The results of this paper once again suggest that the theory of composition operators even when restricted to finite-dimensional model spaces exhibits significant variation from case to case. In particular, the results differ substantially between the cases of finite Blaschke products that vanish at the origin and those that do not. Even within these broad categories, the behavior further varies depending on specific subcases.

The following is a summary of some of the main results concerning finite-dimensional model spaces. These results are established in this paper. However, we reiterate that some of them are derived from [9] and are presented here either verbatim (see also [11]) or in a modified, corrected, or expanded form. In the following, θ denotes a finite Blaschke product, and $\mathcal{Q}_\theta$ refers to the corresponding finite-dimensional model space, with

$$\dim \mathcal{Q}_\theta = \deg \theta.$$

The first row of the following table specifies the conditions imposed on the function θ, while the remaining rows outline the corresponding properties under each condition:

	$\theta(0) \neq 0$	$\theta(0) = 0$
1	$L(Q_\theta)$ is a finite set	$L(Q_\theta) \supseteq \mathbb{C}$, and hence uncountable
2	$L(Q_\theta)$ is a finite cyclic group	$L(Q_\theta) \setminus \mathbb{C}$ is always a group except for one unknown case
3	Every element in $L(Q_\theta)$ is of the form $\varphi(z) = az + b$	Every element in $L(Q_\theta)$ is either a constant or $\frac{az+b}{cz+d}$.
4	In some cases, $L(Q_\theta) = \{z\}$ (cf. Theorem 2.3)	Always $L(Q_\theta) \neq \{z\}$
5	$C_\varphi(Q_\theta) \subseteq Q_\theta \Rightarrow C_\varphi(Q_\theta) = Q_\theta$	$C_\varphi(Q_\theta) \subseteq Q_\theta \Rightarrow C_\varphi(Q_\theta) = Q_\theta$ only when $\theta(z) = z$

These results, along with the techniques used to obtain them, raise several questions of general interest. We conclude the paper by highlighting two particularly intriguing ones: The first question concerns the classification of finite-dimensional spaces that are invariant under composition operators. Specifically, given a finite-dimensional subspace $\mathcal{S} \subseteq H^2(\mathbb{D})$, classify all analytic self-maps $\varphi \in \mathcal{S}(\mathbb{D})$ such that

$$C_\varphi \mathcal{S} \subseteq \mathcal{S}.$$

The second question concerns infinite Blaschke products. In the case of finite Blaschke products that do not vanish at the origin, we observed that finite cyclic groups play a central role in characterizing the self-maps that leave the corresponding finite-dimensional model spaces invariant. Some of these results also make use of elementary tools, such as the prime factorization of natural numbers that arise as the cardinality of the zero set of finite Blaschke products. The natural next step is to explore this (or enhanced) phenomenon for model spaces associated with infinite Blaschke products that also do not vanish at the origin.

It is anticipated that group-theoretic tools will again be relevant, though precisely how they will be used remains an open and interesting problem. We expect a deeper interplay between group theory and analytic function theory to be essential in developing a complete understanding.

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